

# EDGE SCALING LIMITS FOR A FAMILY OF NON-HERMITIAN RANDOM MATRIX ENSEMBLES

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**ABSTRACT.** A family of random matrix ensembles interpolating between the GUE and the Ginibre ensemble of  $n \times n$  matrices with iid centered complex Gaussian entries is considered. The asymptotic spectral distribution in these models is uniform in an ellipse in the complex plane, which collapses to an interval of the real line as the degree of non-Hermiticity diminishes. Scaling limit theorems are proven for the eigenvalue point process at the rightmost edge of the spectrum, and it is shown that a non-trivial transition occurs between Poisson and Airy point process statistics when the ratio of the axes of the supporting ellipse is of order  $n^{-1/3}$ . In this regime, the family of limiting probability distributions of the maximum of the real parts of the eigenvalues interpolates between the Gumbel and Tracy-Widom distributions.

## 1. INTRODUCTION OF THE MODEL

**1.1. Introduction.** The local statistics at the edge of the spectrum are of fundamental interest in the study of random matrix models. For the Gaussian unitary ensemble (GUE), defined as the probability measure

$$(1.1) \quad d\mathbb{P}_n^1(H) = \left( \frac{n^{n^2}}{2^n \pi^{n^2}} \right)^{1/2} \exp \left\{ -\frac{n}{2} \operatorname{Tr} H^2 \right\} dH$$

on the space  $\mathcal{H}_n \cong \mathbb{R}^{n^2}$  of Hermitian matrices, where  $dH$  denotes Lebesgue measure, the celebrated work of Tracy and Widom, [14], shows that the distribution of the largest eigenvalue, properly rescaled, converges to  $F_{TW}$ , the Tracy-Widom distribution, which is expressible in terms of a solution to the Painlevé II differential equation. Their results have been generalized to other models and it is conjectured that  $F_{TW}$  is the universal limiting extreme value law for large classes of random matrix ensembles. This universality conjecture is supported by the fact that the Tracy-Widom distribution has appeared in a number of seemingly unrelated contexts, see for instance [1], [8] and [9]; in some sense it is a natural extreme value distribution.

In contrast, the classical extreme value theorem for a set of *independent* identically distributed (iid) random variables, say  $X_i \in N(0, 1/2)$ , states that the distribution

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of their appropriately shifted and rescaled maximum converges to the *Gumbel* distribution,  $F_G(x) := e^{-e^{-x}}$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \frac{\max_{1 \leq i \leq n} \{X_i\} - c_n}{a_n} \leq t \right] = F_G(t),$$

where

$$c_n = \sqrt{\log n} - \frac{\log(4\pi \log n)}{4\sqrt{\log n}},$$

and

$$a_n = \frac{1}{2\sqrt{\log n}}.$$

Although the edge behaviours of these models have received much less attention in the literature, there are random matrix ensembles without symmetry conditions imposed so that the limiting spectral density is no longer supported on a line, but some other set in the complex plane. The simplest example of such a model is the classical *Ginibre ensemble* of matrices with independent centered complex Gaussian entries of variance  $n^{-1}$ , introduced in [6], for which the asymptotic spectral distribution is uniform in (and vanishes outside) the unit disc. There are several possible notions of “extremeness” that might be considered in this model; Rider [12] shows that the spectral radius, that is, the maximum absolute value of the eigenvalues, is asymptotically Gumbel-distributed after a shifting and rescaling much the same as for the case of extremes of independent random variables. Intuitively, this can be understood as reflecting the fact that the few eigenvalues with largest modulus do not in general lie close to or feel the repulsion of each other, and thus behave as if they were independent. Rider’s method relies heavily on the radial symmetry of the problem; a closely related problem which generalizes easier to the family of models to be considered here is that of determining the distribution of the maximum of the *real parts* of the eigenvalues instead of their absolute values.

**1.2. The ellipse ensemble.** The fundamental difference between the edge statistics of the GUE and those of the Ginibre ensemble corresponds to the distinction between random matrix and Poisson statistics which has attracted a great deal of attention from the mathematical physics community in connection with quantum chaos. In particular, it is illuminating to study families of models that exhibit a transition between the two types of behaviour, see e.g. [10]. Based on the observation that a Ginibre matrix  $A$  can equivalently be considered as a sum  $A = (H_1 + iH_2)/\sqrt{2}$  where  $H_1$  and  $H_2$  are two independent GUEs, a natural family of random matrix ensembles interpolating between the GUE and Ginibre ensembles can be defined as the distribution of

$$(1.2) \quad A = \sqrt{\frac{(1+\tau)}{2}} H_1 + i \sqrt{\frac{(1-\tau)}{2}} H_2,$$

for  $\tau \in [0, 1]$ . The parameter  $\tau$  can thus be thought of as determining the extent to which  $A$  fails to be Hermitian. In [4], the authors define this *ellipse ensemble*

explicitly as the probability measure  $\mathbb{P}_n^\tau$  on the space  $\mathcal{M}_n$  of complex  $n \times n$  matrices given, for  $\tau < 1$ , by

$$(1.3) \quad d\mathbb{P}_n^\tau(A) = \left( \frac{n}{\pi\sqrt{(1-\tau^2)}} \right)^{n^2} \exp \left\{ -\frac{n}{(1-\tau^2)} \operatorname{Tr}(AA^\dagger - \tau \operatorname{Re} A^2) \right\} dA,$$

where  $dA$  is Lebesgue measure on  $\mathcal{M}_n \cong \mathbb{R}^{2n^2}$ . It is easy to see that the cases  $\tau = 0$  and  $\tau \rightarrow 1$  correspond to the Ginibre and Gaussian unitary ensembles respectively. The successful analysis of this model depends crucially on the fact that, for every  $\tau \in (0, 1)$ , the induced eigenvalue measure

$$(1.4) \quad d\mathbb{P}_n^{\tau'}(\zeta_1, \dots, \zeta_n) = \frac{1}{\mathcal{Z}_N^\tau} \prod_{1 \leq j < k \leq n} |\zeta_k - \zeta_j|^2 \exp \left\{ -\frac{n}{(1-\tau^2)} \sum_{j=1}^n (|\zeta_j|^2 - \tau \operatorname{Re} \zeta_j^2) \right\} d^n \zeta_j,$$

where  $\mathcal{Z}_N^\tau$  is a normalizing constant, can be considered as the distribution of a point process  $Z_n^\tau$  on  $\mathbb{R}^2 (\cong \mathbb{C})$  with determinantal correlation functions given by the correlation kernel

$$(1.5) \quad K_n^\tau((\xi_1, \eta_1), (\xi_2, \eta_2)) = \frac{n}{\sqrt{\pi(1-\tau^2)}} \sum_{k=0}^{n-1} \tau^k h_k \left( \sqrt{\frac{n}{2\tau}} \zeta_1 \right) h_k \left( \sqrt{\frac{n}{2\tau}} \bar{\zeta}_2 \right) \exp \left\{ -\frac{n}{2} \left( \frac{\xi_1^2 + \xi_2^2}{(1+\tau)} + \frac{\eta_1^2 + \eta_2^2}{(1-\tau)} \right) \right\}.$$

Here  $\zeta_j = \xi_j + i\eta_j$ , and  $h_k$  are the normalized Hermite polynomials, that is, the orthonormal polynomials with respect to the measure  $e^{-x^2} dx$  on  $\mathbb{R}$ . (See Section 2.1 for definitions related to determinantal point processes.) Furthermore, the rescaled Hermite polynomials appearing in (1.5) are orthogonal in the whole complex plane with respect to the corresponding exponential weight. The orthogonal polynomial techniques employed in the analysis of the GUE therefore become available for the ellipse ensemble as well, although considerable technical difficulties remain, since the Christoffel-Darboux formula no longer applies to simplify the sum of products of *rescaled* Hermite polynomials. It is proven in [4] e.g. that for fixed  $\tau \neq 1$ , the limiting eigenvalue density is constant in the ellipse  $\{(x, y) : x^2/(1+\tau)^2 + y^2/(1-\tau)^2 \leq 1\}$ , where  $z = x + iy$ . The authors also discover a regime of “weak non-Hermiticity”, when  $\tau = \tau_n = 1 - \alpha^2/(2n)$  for a new parameter  $0 < \alpha < \infty$ , in which a transition occurs in the local bulk statistics between the sine kernel behaviour of the GUE and the scaled correlation kernel of the Ginibre ensemble. In [5], it is shown heuristically that in this regime the local statistics near the rightmost edge of the spectrum are given by a correlation kernel which factorizes as the Airy kernel in the  $\xi$ -variables multiplied by a Gaussian weight in the  $\eta$ -direction; thus the model essentially behaves like the GUE at the edge in this regime.

However, as the present work will show, there is another transitional regime, for  $(1 - \tau_n) \sim n^{-1/3}$ , when the *edge* statistics interpolate between those of the GUE and

the Ginibre ensemble. Let  $\sigma_n = n^{1/6} \sqrt{(1 - \tau_n)}$ . It is proven in Theorem 2.5 that if  $\sigma_n \rightarrow 0$ , the rescaled eigenvalues near the rightmost edge of the spectrum converge to a point process on  $\mathbb{R}^2$  which can be interpreted as the Airy point process in the  $x$ -direction with each particle subject to an iid centered Gaussian displacement in the  $y$ -direction. Furthermore, the distribution of the maximum of the real parts of the eigenvalues converges to the Tracy-Widom distribution. The process is thus essentially one-dimensional and coincides with the corresponding point process at the edge of the GUE. (This is what is found in [5] for the particular choice  $\sigma_n \propto n^{-1/3}$ .) If  $\sigma_n \rightarrow \infty$  the edge point process converges to a Poisson process on  $\mathbb{R}^2$  with intensity  $\pi^{-1/2} e^{-x-y^2}$ , and in addition the maximum of the real parts of the eigenvalues converges in distribution to a Gumbel random variable. In the intermediate regime when  $\sigma_n \rightarrow \sigma \in \mathbb{R}^+$  a new non-trivial interpolating point process on  $\mathbb{R}^2$  arises in the  $n \rightarrow \infty$  limit. It is again a determinantal point process with a rightmost particle almost surely, but the correlation kernel no longer factorizes.

The edge behaviour in the various regimes can be interpreted as follows: When  $\sigma_n \rightarrow \infty$ , the supporting ellipse collapses to an interval of the real line so slowly that the imaginary parts of the eigenvalues near the edge are of order much greater than the spacing of their real parts. Therefore the eigenvalues are not close and do not interact, so in the appropriate scaling limit they behave as a Poisson process. On the other hand, when  $\sigma_n \rightarrow 0$ , the imaginary parts of the eigenvalues near the edge are negligible compared to the spacing of their real parts; hence they behave essentially like the particles in the Airy point process with independent fluctuations in the  $y$ -direction. In contrast, the case  $\sigma_n \rightarrow \sigma$  corresponds precisely to the imaginary parts of the eigenvalues being of the same order of magnitude as the spacings of their real parts, and there is a non-trivial interaction between eigenvalues even though their displacement away from the real axis is no longer negligible.

The proof relies in an essential way on finding, in the absence of a Christoffel-Darboux formula, a manageable representation of the sum of products of Hermite polynomials in (1.5). This is provided by the double integral formula of Lemma 4.1. The asymptotics of the correlation kernel can then be analyzed by means of saddle point techniques.

Although a very specific family of models, particularly accessible to analysis, is studied here, it is reasonable to conjecture, in analogy with the universality conjectures for random matrix ensembles with purely real spectrum, that the type of transition and scale at which it occurs has a universal character that should be found in much greater generality for random matrix models in which the support of the spectral density collapses to a one-dimensional subset of the complex plane. Examples of other non-Hermitian ensembles are provided by the model introduced in [15] and analyzed rigorously in [7].

## 2. FORMULATION OF RESULTS

**2.1. Preliminaries on determinantal point processes.** For a comprehensive account of the theory of point processes, see [2]. The material relating to determinantal processes can be found e.g. in [11].

Let  $\Lambda$  be a complete separable metric space. A Borel measure  $\mu$  on  $\Lambda$  is a *boundedly finite simple counting measure* if it takes non-negative integer values on bounded sets and  $\mu(\{x\}) \in \{0, 1\}$  for every  $x \in \Lambda$ . The set  $\mathcal{N}(\Lambda)$  of all boundedly finite simple counting measures on  $\Lambda$  can be identified with the family of sets  $x = \{x_i\} \subset \Lambda$  such that  $x_i \neq x_j$  if  $i \neq j$  and  $|x \cap B| < \infty$  for every bounded  $B \subset \Lambda$ . Define the smallest  $\sigma$ -algebra on  $\mathcal{N}(\Lambda)$  such that the mappings  $\mathcal{N}(\Lambda) \ni \mu \mapsto \mu(A)$  are measurable for each Borel set  $A \in \Lambda$ , making  $\mathcal{N}(\Lambda)$  a measurable space. A *point process*  $X = \{x_j\}$  on  $\Lambda$  is a random element of  $\mathcal{N}(\Lambda)$ .

Let  $\lambda$  be a reference measure on  $\Lambda$  (in our case,  $\Lambda$  will be  $\mathbb{R}$  or  $\mathbb{R}^2$  and  $\lambda$  Lebesgue measure). If  $X$  is a point process on  $\Lambda$  and, for some  $n \geq 1$ , there exists a measurable function  $\rho_n : \Lambda^n \rightarrow \mathbb{R}$  such that for every bounded measurable function  $\varphi : \Lambda^n \rightarrow \mathbb{R}$ ,

$$(2.1) \quad \mathbb{E} \left[ \sum_{x_{k_j} \in X} \varphi(x_{k_1}, \dots, x_{k_n}) \right] = \int_{\Lambda^n} \varphi(\xi_1, \dots, \xi_n) \rho_n(\xi_1, \dots, \xi_n) d^n \lambda(\xi),$$

then  $\rho_n$  is called an *n-point correlation function* of  $X$ . (The sum on the left hand side of (2.1) is over all  $n$ -tuples of distinct points in  $X$ , including permutations.) It can be shown that if  $X$  has correlation functions  $\rho_n$  for every  $n \geq 1$  (and the product over an empty index set is 1 by definition), then

$$(2.2) \quad \mathbb{E} \left[ \prod_j (1 + \phi(x_j)) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(\xi_j) \rho_n(\xi_1, \dots, \xi_n) d^n \lambda(\xi)$$

for every bounded measurable function  $\phi$  with bounded support. (Here the product on the left hand side is over all particles in the point process; since there are only finitely many particles in each bounded set the product is finite for each realization of  $X$ ).

A point process for which all correlation functions exist and are of the form

$$\rho_n(\xi_1, \dots, \xi_n) = \det (K(\xi_i, \xi_j))_{i,j=1}^n$$

for some measurable function  $K : \Lambda^2 \rightarrow \mathbb{C}$  (a *correlation kernel*) is called a *determinantal (point) process*. The correlation kernel  $K$  is said to be *Hermitian* if

$$K(\xi_2, \xi_1) = \overline{K(\xi_1, \xi_2)}$$

for all  $(\xi_1, \xi_2) \in \Lambda^2$ .

Weak convergence in  $\mathcal{N}(\Lambda)$  of a sequence  $\{X_n\}$  of point processes to a limit  $X$  is equivalent to weak convergence of the finite dimensional distributions, meaning that

for any finite family  $\{A_i\}_{i=1}^k$  of bounded disjoint Borel sets in  $\Lambda$  and integers  $m_i$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left( \bigcap_{i=1}^k [|X_n \cap A_i| = m_i] \right) = \mathbb{P} \left( \bigcap_{i=1}^k [|X \cap A_i| = m_i] \right).$$

Consider a determinantal point process  $Z = \{z_j\} = \{(x_j, y_j)\}$  on  $\mathbb{R}^2$ . If there is a  $t \in \mathbb{R}$  such that  $\mathbb{E}[|Z \cap ((t, \infty) \times \mathbb{R})|] < \infty$ ,  $Z$  is said to have a *rightmost* or *last* particle almost surely. The *last particle distribution function*,  $F$ , of  $Z$  is then defined as

$$(2.3) \quad F(t) = \mathbb{P}[|Z \cap ((t, \infty) \times \mathbb{R})| = 0].$$

To prove the limit theorems of this paper we will appeal to the following lemma, which is a fairly standard result. For completeness, a proof is given in an appendix.

**Lemma 2.1.** *For every positive integer  $n$ , let  $Z_n$  be a determinantal point process on  $\mathbb{R}^2$  with Hermitian correlation kernel  $K_n$ . Suppose that  $K_n(\zeta_1, \zeta_2) \rightarrow K(\zeta_1, \zeta_2)$  as  $n \rightarrow \infty$  for every  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$ , and that there is a measurable function  $B : \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $B(\zeta) \geq |K_n(\zeta, \zeta)|$  for every  $n$ , such that for each  $\xi_0 \in \mathbb{R}$*

$$\int_{(\xi_0, \infty) \times \mathbb{R}} B(\zeta) d\zeta = C_{\xi_0} < \infty.$$

*Then  $Z_n$  converges weakly as  $n \rightarrow \infty$  to a determinantal point process  $Z$  with correlation kernel  $K$ ,  $Z_n$  has a last particle almost surely with distribution*

$$(2.4) \quad F_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_{((t, \infty) \times \mathbb{R})^r} \det(K_n(\zeta_j, \zeta_k))_{j,k=1}^r d^r \zeta,$$

*and  $F_n$  converges to the last particle distribution  $F$  of  $Z$  as  $n \rightarrow \infty$ .*

**2.2. The interpolating process.** The *Airy kernel* is the function  $K_A : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$K_A(\xi_1, \xi_2) = \frac{\text{Ai}(\xi_1) \text{Ai}'(\xi_2) - \text{Ai}'(\xi_1) \text{Ai}(\xi_2)}{\xi_1 - \xi_2}$$

if  $\xi_1 \neq \xi_2$  and by continuity if  $\xi_1 = \xi_2$ . Here  $\text{Ai} : \mathbb{R} \rightarrow \mathbb{R}$  is the  $C^\infty$  *Airy function*,

$$\text{Ai}(t) = \frac{1}{2\pi} \int_{\gamma} e^{\frac{i}{3}u^3 + iut} du \in \mathbb{R},$$

and  $\gamma$  is the contour  $t \mapsto t + i\delta$  for some  $\delta > 0$ , independently of the choice of  $\delta$  by Cauchy's theorem. We will use the double integral representation

$$(2.5) \quad K_A(\xi_1, \xi_2) = \frac{1}{4\pi^2} \iint_{\gamma, \gamma} \frac{e^{\frac{i}{3}v^3 + i\xi_2 v + \frac{i}{3}u^3 + i\xi_1 u}}{i(u+v)} du dv$$

of the Airy kernel. It is well known (see [14] and [3]) that  $K_A$  defines a trace class integral operator on  $L^2(t, \infty)$  for any real  $t$  and that it is the correlation kernel of a determinantal point process  $X_A$  on  $\mathbb{R}$ , the *Airy point process*.  $X_A$  has a last particle

almost surely, the distribution of which is known as the *Tracy-Widom distribution*, given by the Fredholm determinant

$$F_{TW}(t) := \det(I - K_A)_{L^2(t, \infty)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{(t, \infty)^k} \det(K_A(\xi_i, \xi_j))_{i,j=1}^k d^k \xi.$$

Alternatively,

$$(2.6) \quad F_{TW}(t) = \exp \left\{ - \int_t^{\infty} (x-t) q^2(x) dx \right\},$$

where  $q$  is the unique solution to the Painlevé II equation

$$q''(t) = tq(t) + 2q(t)^3$$

with the asymptotic behaviour

$$q(t) \sim \text{Ai}(t) \text{ as } t \rightarrow \infty.$$

We now construct a point process  $Z_A$  on  $\mathbb{R}^2$  which, intuitively, to each realization  $\{x_i\}$  of  $X_A$  associates a realization  $\{(x_i, y_i)\}$  of  $Z_A$ , where the  $y_i$  are picked independently from  $N(0, 1/2)$ . More precisely, for every  $k \geq 1$ ,  $Z_A$  should have correlation functions

$$(2.7) \quad \begin{aligned} \rho_k(\zeta_1, \dots, \zeta_k) &:= \det(K_A(\xi_i, \xi_j))_{i,j=1}^k \prod_{i=1}^k \frac{e^{-\eta_i^2}}{\sqrt{\pi}} \\ &= \det \left( \frac{e^{-\frac{1}{2}(\eta_i^2 + \eta_j^2)}}{\sqrt{\pi}} K_A(\xi_i, \xi_j) \right)_{i,j=1}^k, \end{aligned}$$

making  $Z_A$  a determinantal process as well.

A Poisson process  $Z_P$  on  $\mathbb{R}^2$  with intensity  $\pi^{-1/2} e^{-\xi - \eta^2}$  can be viewed as a degenerate determinantal point process with correlation kernel

$$(2.8) \quad M_P(\zeta_1, \zeta_2) = \delta_{\zeta_1 \zeta_2} \frac{e^{-\xi_1 - \eta_1^2}}{\sqrt{\pi}} = \begin{cases} \pi^{-1/2} e^{-\xi_1 - \eta_1^2} & \text{if } \zeta_1 = \zeta_2 \\ 0 & \text{otherwise.} \end{cases}$$

To stress the analogy with the construction above, define

$$(2.9) \quad M_{P1}(\zeta_1, \zeta_2) = \frac{e^{-\frac{1}{2}(\eta_1^2 + \eta_2^2)}}{\sqrt{\pi}} K_P(\xi_1, \xi_2),$$

where  $K_P(\xi_1, \xi_2) = \delta_{\xi_1 \xi_2} e^{-\xi_1}$  is the correlation kernel of a Poisson process  $X_P$  on  $\mathbb{R}$  with intensity  $e^{-\xi}$ .  $X_P$  too has a last particle almost surely, which is Gumbel-distributed, since

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(t, \infty)^n} \det(K_P(\xi_i, \xi_j))_{i,j=1}^n d^n \xi \\ = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(t, \infty)^n} \prod_{i=1}^n e^{-\xi_i} d^n \xi = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (e^{-t})^n = F_G(t). \end{aligned}$$

For any  $n \geq 1$ ,

$$\det(M_P(\zeta_j, \zeta_k))_{j,k=1}^n = \det(M_{P_1}(\zeta_j, \zeta_k))_{j,k=1}^n \text{ a.e. in } \mathbb{R}^{2n},$$

so  $M_{P_1}$  can equivalently be chosen as a correlation kernel of  $Z_P$ . Similarly,

$$(2.10) \quad M_{P_2}(\zeta_1, \zeta_2) = \delta_{\eta_1 \eta_2} \frac{e^{-\eta_1^2 - \frac{1}{2}(\xi_1^2 + \xi_2^2)}}{\sqrt{\pi}}$$

is also an equivalent correlation kernel of  $Z_P$ . The following proposition sums up these observations.

**Proposition 2.2.** *The determinantal point processes  $Z_P$  and  $Z_A$  on  $\mathbb{R}^2$  with correlation kernels  $M_P$  (or equivalently  $M_{P_1}$  or  $M_{P_2}$ ) and*

$$(2.11) \quad M_A(\zeta_1, \zeta_2) = \frac{e^{-\frac{1}{2}(\eta_1^2 + \eta_2^2)}}{\sqrt{\pi}} K_A(\xi_1, \xi_2)$$

*respectively, both have last particles almost surely, with distribution functions  $F_G$  and  $F_{TW}$  respectively.*

*Proof.* If  $K$  is a Hermitian correlation kernel of a determinantal point process on  $\mathbb{R}$  satisfying

$$\int_t^\infty K(\xi, \xi) d\xi < \infty,$$

it is immediate that  $M(\zeta_1, \zeta_2) = \pi^{-1/2} e^{-\frac{1}{2}(\eta_1^2 + \eta_2^2)} K(\xi_1, \xi_2)$  satisfies the corresponding conditions of Lemma A.2. Since the correlation functions factorize by (2.7), the  $\eta_i$ -variables may be integrated out in the expression (A.2) for the last particle distribution.  $\square$

The following theorem asserts the existence of a family of determinantal point processes  $\{Z_\sigma\}_{\sigma>0}$  on  $\mathbb{R}^2$  which, appropriately rescaled, interpolates between  $Z_A$  and  $Z_P$ .

**Theorem 2.3.** (Interpolating process) *For each  $\sigma \in [0, \infty)$  there exists a determinantal point process*

$$Z_\sigma = \{(x_j, y_j)\}$$

*on  $\mathbb{R}^2$  with correlation kernel*

$$(2.12) \quad M_\sigma(\zeta_1, \zeta_2) = \frac{1}{4\pi^{5/2}} \int_{\gamma_1} \int_{\gamma_2} \frac{e^{-\frac{1}{2}(\sigma v - \eta_2)^2 + \frac{i}{3}v^3 + i\xi_2 v - \frac{1}{2}(\sigma u + \eta_1)^2 + \frac{i}{3}u^3 + i\xi_1 u}}{i(u+v)} du dv,$$

*where  $\zeta_j = (\xi_j, \eta_j)$  and  $\gamma_j$  is the contour  $t \mapsto \gamma_j(t) = t + i\delta_j$ , independently of the choice of  $\delta_j > 0$ .*

*Define the rescaled point process*

$$\tilde{Z}_\sigma = \left\{ \left( \frac{x_j - c_\sigma}{a_\sigma}, \frac{y_j}{b_\sigma} \right) \right\},$$

*where*

$$a_\sigma = \frac{\sigma}{\sqrt{6 \log \sigma}},$$



$$b_\sigma = \frac{\sigma^{3/2}}{(6 \log \sigma)^{1/4}},$$

and

$$c_\sigma = a_\sigma \left( 3 \log \sigma - \frac{5}{4} \log(6 \log \sigma) - \log(2\pi) \right).$$

The processes  $Z_\sigma$ , appropriately rescaled, interpolate between  $Z_A$  and  $Z_P$  in the sense that  $Z_0 = Z_A$  and  $\tilde{Z}_\sigma$  converges weakly in  $\mathcal{N}(\mathbb{R}^2)$  to  $Z_P$  as  $\sigma \rightarrow \infty$ . For any fixed  $\sigma$ ,  $Z_\sigma$  has a last particle almost surely, with distribution function

$$\begin{aligned} F_\sigma(t) &= \mathbb{P}_\sigma [|Z_\sigma \cap ((t, \infty) \times \mathbb{R})| = 0] \\ &= \det(I - M_\sigma)_{L^2((t, \infty) \times \mathbb{R})} \\ (2.13) \quad &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_{((t, \infty) \times \mathbb{R})^r} \det(M_\sigma(\zeta_j, \zeta_k))_{j,k=1}^r d^r \xi d^r \eta, \end{aligned}$$

and furthermore

$$(2.14) \quad F_\sigma(c_\sigma + a_\sigma t) \rightarrow F_G(t) \text{ as } \sigma \rightarrow \infty.$$

**Remark 2.4.** To the author's knowledge, the processes  $Z_\sigma$  have not been studied previously. In particular, we are not aware of any generalization to  $\sigma > 0$  of the representation (2.6), for the case  $\sigma = 0$ , of the Fredholm determinant (2.13).

The proof is deferred to Section 3.

**2.3. Edge scaling limits.** This section contains the statement of the main result, giving scaling limits around the rightmost particle of the eigenvalue point process  $Z_n^{\tau_n} = \{z_j\}_{j=1}^n = \{(x_j, y_j)\}_{j=1}^n$  of the ellipse ensemble, defined formally as the measurable function

$$Z_n^{\tau_n} : (\mathcal{M}_n, \mathbb{P}_n^{\tau_n}) \rightarrow \mathcal{N}(\mathbb{R}^2)$$

mapping  $A \in \mathcal{M}_n$  to the set  $\{z_j\}_{j=1}^n$  of eigenvalues of  $A$  (with probability one, there are  $n$  distinct (complex) eigenvalues and we assume this to be the case). Recall that  $Z_n^{\tau_n}$  is determinantal with correlation kernel  $K_n^{\tau_n}$ .

**Theorem 2.5.** *Let  $\{\tau_n\}_{n=1}^\infty \subseteq [0, 1)$  be given and put  $\sigma_n = n^{1/6} \sqrt{1 - \tau_n}$ . For the choices of scaling parameters  $\tilde{a}_n$ ,  $\tilde{b}_n$  and  $\tilde{c}_n$  specified below, define the rescaled edge eigenvalue point process*

$$\tilde{Z}_n^{\tau_n} = \left\{ \left( \frac{x_j - \tilde{c}_n}{\tilde{a}_n}, \frac{y_j}{\tilde{b}_n} \right) \right\}_{j=1}^n,$$

and let

$$\begin{aligned} F_n^{\tau_n}(t) &= \mathbb{P}_n^{\tau_n} \left[ \left| \tilde{Z}_n^{\tau_n} \cap ((t, \infty) \times \mathbb{R}) \right| = 0 \right] \\ &= \mathbb{P}_n^{\tau_n} \left[ \frac{\max_{1 \leq j \leq n} \{x_j\} - \tilde{c}_n}{\tilde{a}_n} \leq t \right] \end{aligned}$$

be the last particle distribution of  $\tilde{Z}_n^{\tau_n}$ .

(i) Suppose  $\sigma_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Choose

$$(2.15) \quad \tilde{a}_n = \hat{\tau}_n^{1/2} \frac{\sigma_n n^{-2/3}}{\sqrt{6 \log \sigma_n}},$$

$$(2.16) \quad \tilde{b}_n = \hat{\tau}_n^{-1/4} \frac{\sigma_n^{5/2} n^{-2/3}}{(6 \log \sigma_n)^{1/4}},$$

and

$$(2.17) \quad \tilde{c}_n = 2\hat{\tau}_n + \tilde{a}_n \left( 3 \log \sigma_n - \frac{5}{4} \log(6 \log \sigma_n) - \log(2\pi \hat{\tau}_n^{3/4}) \right),$$

where  $\hat{\tau}_n := (1 + \tau_n)/2$ . Then  $\tilde{Z}_n^{\tau_n}$  converges weakly to  $Z_P$  and  $F_n^{\tau_n}(t)$  converges to  $F_G(t)$  as  $n \rightarrow \infty$ .

(ii) Suppose  $\sigma_n \rightarrow \sigma \in [0, \infty)$  as  $n \rightarrow \infty$ . Choose  $\tilde{a}_n = n^{-2/3}$ ,  $\tilde{b}_n = \sigma_n n^{-2/3}$  and  $\tilde{c}_n = (1 + \tau_n)$ . Then  $\tilde{Z}_n^{\tau_n}$  converges weakly to  $Z_\sigma$  and  $F_n^{\tau_n}(t)$  converges to  $F_\sigma(t)$  as  $n \rightarrow \infty$ .

In particular, if  $\sigma = 0$ ,  $\tilde{Z}_n^{\tau_n}$  converges weakly to  $Z_A$  and  $F_n^{\tau_n}(t)$  converges to  $F_{TW}(t)$ .

**Remark 2.6.** Although the case  $\tau_n = 0$  has not been explicitly included in the calculations, a similar (simpler) argument can be made in this case (see the comment following (4.10)); as can be expected from the fact that the ellipse ensemble for constant  $\tau \in (0, 1)$  is just a trivial rescaling of the Ginibre ensemble (see [4]), formally inserting  $\tau_n = 0$  in (2.15) through (2.17) indeed gives the correct limit theorem for the pure Ginibre ensemble as well.

**Remark 2.7.** In view of the scaling limits in Theorems 2.3 and 2.5 (ii), the scaling limit in Theorem 2.5 (i) is the only reasonable candidate (as long as  $\tau_n \rightarrow 1$ ); together these theorems in essence assert that  $\tilde{Z}_n^{\tau(\sigma)}$  converges to  $Z_P$  regardless of whether  $\sigma$  tends to infinity with  $n$  or the  $n$  limit is taken first.

### 3. PROOF OF THEOREM 2.3

The asymptotic notation defined in the first paragraph of Section 4 is used in this section as well, with  $n$  replaced by  $\sigma$ .

*Proof of Theorem 2.3.* The existence of  $Z_\sigma$ , and the fact that it has a last particle with distribution (2.13), follows by Lemma 2.1 from the convergence of  $Z_n^{\tau_n}$  when  $(\tau_n)_{n=1}^\infty$  is a sequence such that  $n^{1/6} \sqrt{(1 - \tau_n)} \rightarrow \sigma$  as  $n \rightarrow \infty$ . This is established in the proof of Theorem 2.5. Note that  $Z_0$  coincides with  $Z_A$ , by (2.5).

To prove weak convergence of  $\tilde{Z}_\sigma$  to  $Z_P$  using Lemma 2.1, we first prove point-wise convergence of the correlation kernels. From the definition of correlation functions (2.1), it is clear that the rescaled point process  $\tilde{Z}_\sigma$  is determinantal with correlation kernel

$$(3.1) \quad \tilde{M}_\sigma(\zeta_1, \zeta_2) := a_\sigma b_\sigma M_\sigma((c_\sigma + a_\sigma \xi_1, b_\sigma \eta_1), (c_\sigma + a_\sigma \xi_2, b_\sigma \eta_2)).$$

We fix  $(\zeta_1, \zeta_2) = ((\xi_1, \eta_1), (\xi_2, \eta_2)) \in \mathbb{R}^4$  and calculate the  $\sigma \rightarrow \infty$  limit of  $\tilde{M}_\sigma$  by a saddle point argument. Let

$$(3.2) \quad f_\sigma(u) = -\frac{1}{2}(\sigma u + \eta'_1)^2 + \frac{i}{3}u^3 + i\xi'_1 u,$$

and

$$(3.3) \quad g_\sigma(v) = -\frac{1}{2}(\sigma v - \eta'_2)^2 + \frac{i}{3}v^3 + i\xi'_2 v,$$

where  $\xi'_j = c_\sigma + a_\sigma \xi_j$  and  $\eta'_j = b_\sigma \eta_j$ . Choose the saddle points  $u_0$  of  $f_\sigma$  and  $v_0$  of  $g_\sigma$  satisfying

$$u_0 = -\frac{\eta'_1}{\sigma} + \frac{i\xi'_1}{\sigma^2} + \frac{2i\eta'^2_1}{\sigma^4} + \frac{2\xi'_1\eta'_1}{\sigma^5} + \mathcal{O}\left(\frac{1}{\sigma^{5/2}(\log \sigma)^{3/4}}\right),$$

and

$$v_0 = \frac{\eta'_2}{\sigma} + \frac{i\xi'_2}{\sigma^2} + \frac{2i\eta'^2_2}{\sigma^4} - \frac{2\xi'_2\eta'_2}{\sigma^5} + \mathcal{O}\left(\frac{1}{\sigma^{5/2}(\log \sigma)^{3/4}}\right),$$

respectively. Note that

$$g''_\sigma(v_0) = -\sigma^2 + 2iv_0 = -\sigma^2 \left(1 + \mathcal{O}\left(\frac{1}{\sigma^{3/2}(\log \sigma)^{1/4}}\right)\right)$$

and similarly  $f''_\sigma(u_0) = -\sigma^2 + 2iu_0$ . After the change of variables  $s = \sigma(u - u_0)$ ,  $t = \sigma(v - v_0)$ , and choosing  $\delta_1 = \text{Im}(u_0)$  and  $\delta_2 = \text{Im}(v_0)$  so that the new contours of integration become the real axis, the rescaled kernel can be written

$$(3.4) \quad \tilde{M}_\sigma(\zeta_1, \zeta_2) = \frac{a_\sigma b_\sigma}{4\pi^{5/2}\sigma^2} \iint_{\mathbb{R}^2} \frac{e^{f_\sigma(s/\sigma+u_0)+g_\sigma(t/\sigma+v_0)}}{i(t/\sigma+u_0+s/\sigma+v_0)} ds dt.$$

The main contribution comes from near the saddle points; let  $I = (-r_0, r_0)$  for some  $1 \ll r_0 \ll \sigma^{3/4}(\log \sigma)^{1/8}$  and note that

$$(3.5) \quad \left| \frac{a_\sigma b_\sigma}{4\pi^{5/2}\sigma^2} \iint_{\mathbb{R} \times (\mathbb{R} \setminus I)} \frac{e^{f_\sigma(s/\sigma+u_0)+g_\sigma(t/\sigma+v_0)}}{i(t/\sigma+u_0+s/\sigma+v_0)} ds dt \right| \\ \leq \frac{a_\sigma b_\sigma}{4\pi^{5/2}\sigma^2(\delta_1 + \delta_2)} \left| \int_{\mathbb{R}} e^{f_\sigma(s/\sigma+u_0)} ds \right| \left| \int_{\mathbb{R} \setminus I} e^{g_\sigma(t/\sigma+v_0)} dt \right|.$$

Now

$$g_\sigma(t/\sigma + v_0) \\ = g_\sigma(v_0) + t^2 \left( -\frac{1}{2} + \frac{iv_0}{\sigma^2} \right) + t \left( -\sigma v_0 + \eta'_2 + \frac{iv_0^2}{\sigma} + \frac{i\xi'_2}{\sigma} \right) + \frac{it^3}{3\sigma^3},$$

so putting

$$\epsilon_1 = \text{Re} \left( \frac{iv_0}{\sigma^2} \right) = \mathcal{O} \left( \frac{1}{\sigma^{3/2}(\log \sigma)^{1/4}} \right)$$

and

$$\epsilon_2 = \operatorname{Re} \left( -\sigma v_0 + \eta'_2 + \frac{iv_0^2}{\sigma} + \frac{i\xi'_2}{\sigma} \right) = \mathcal{O} \left( \frac{(\log \sigma)^{1/4}}{\sigma^{3/2}} \right)$$

gives

$$\left| \int_{\mathbb{R} \setminus I} e^{g_\sigma(t/\sigma + v_0)} dt \right| \leq |e^{g_\sigma(v_0)}| \left| \int_{\mathbb{R} \setminus I} e^{t^2(-\frac{1}{2} - \epsilon_1) + \epsilon_2 t} dt \right| \leq C \frac{e^{-r_0^2/3}}{r_0} |e^{g_\sigma(v_0)}|.$$

By the same argument for  $f_\sigma$ , and (3.5), it follows that

$$(3.6) \quad \left| \frac{a_\sigma b_\sigma}{4\pi^{5/2}\sigma^2} \iint_{\mathbb{R}^2 \setminus I^2} \frac{e^{f_\sigma(s/\sigma + u_0) + g_\sigma(t/\sigma + v_0)}}{i(t/\sigma + u_0 + s/\sigma + v_0)} ds dt \right| \\ \leq C_1 \frac{e^{-r_0^2/3}}{r_0} \frac{a_\sigma b_\sigma |e^{g_\sigma(v_0) + f_\sigma(u_0)}|}{\sigma^2(\delta_1 + \delta_2)} \leq C \frac{e^{-r_0^2/3}}{r_0}.$$

Since  $\operatorname{Im}(\sigma t + \sigma^2 u_0 + \sigma s + \sigma^2 v_0) \asymp 2c_\sigma > 0$ , we can write

$$\frac{1}{i(\sigma t + \sigma^2 u_0 + \sigma s + \sigma^2 v_0)} = - \int_0^\infty e^{ip(\sigma(t+s) + \sigma^2(u_0 + v_0))} dp.$$

Therefore, using (3.6) and Fubini's theorem, (3.4) becomes

$$(3.7) \quad \begin{aligned} \tilde{M}_\sigma(\zeta_1, \zeta_2) &= \frac{a_\sigma b_\sigma}{4\pi^{5/2}\sigma^2} \iint_{I^2} \frac{e^{f_\sigma(s/\sigma + u_0) + g_\sigma(t/\sigma + v_0)}}{i(t/\sigma + u_0 + s/\sigma + v_0)} ds dt + o(1) \\ &= \frac{-a_\sigma b_\sigma e^{g_\sigma(v_0) + f_\sigma(u_0)}}{4\pi^{5/2}} \\ &\quad \times \iint_{I^2} \int_0^\infty e^{-\frac{1}{2}(s^2 + t^2) + ip(\sigma(t+s) + \sigma^2(u_0 + v_0))} dp ds dt + o(1) \\ &= \frac{-a_\sigma b_\sigma e^{g_\sigma(v_0) + f_\sigma(u_0)}}{2\pi^{3/2}} \int_0^\infty e^{-\sigma^2 p^2 + i\sigma^2 p(u_0 + v_0)} dp + o(1) \\ &= \frac{-a_\sigma b_\sigma e^{g_\sigma(v_0) + f_\sigma(u_0)}}{2\pi^{3/2} i \sigma^2 (u_0 + v_0)} + o(1), \end{aligned}$$

where the last equality follows from the estimate

$$(3.8) \quad \left| \int_0^\infty e^{-\alpha_\sigma r^2 - (\beta_\sigma + i\gamma_\sigma)r} dr - \frac{1}{\beta_\sigma + i\gamma_\sigma} \right| \\ \leq C \frac{1}{|\beta_\sigma + i\gamma_\sigma|} \left( \left( \frac{\sqrt{\alpha_\sigma}}{\beta_\sigma} \right)^2 + e^{-\beta_\sigma/\sqrt{\alpha_\sigma}} \right),$$

obtained by integration by parts; in this case  $\beta_n/\sqrt{\alpha_n} \asymp \sqrt{6 \log \sigma} \gg 1$ . Factors of the form  $\exp\{F(\xi'_1, \eta'_1) - F(\xi'_2, \eta'_2)\}$  can be freely multiplied to the correlation kernel without changing the correlation functions. Taking  $F(\xi', \eta') = i\xi'\eta'/\sigma - i\eta'^3/3\sigma^3$

and calculating

$$f_\sigma(u_0) = \frac{i\xi'_1\eta'_1}{\sigma} - \frac{\xi_1'^2}{2\sigma^2} - \frac{i\eta_1'^3}{3\sigma^3} - \frac{\xi_1'\eta_1'^2}{\sigma^4} + \mathcal{O}\left(\frac{(\log \sigma)^{3/4}}{\sigma^{3/2}}\right)$$

and

$$g_\sigma(v_0) = -\frac{i\xi'_2\eta'_2}{\sigma} - \frac{\xi_2'^2}{2\sigma^2} + \frac{i\eta_2'^3}{3\sigma^3} - \frac{\xi_2'\eta_2'^2}{\sigma^4} + \mathcal{O}\left(\frac{(\log \sigma)^{3/4}}{\sigma^{3/2}}\right),$$

Equation (3.7) gives the equivalent kernel

$$\begin{aligned} M_\sigma^*(\zeta_1, \zeta_2) &= e^{F(\xi'_1, \eta'_1) - F(\xi'_2, \eta'_2)} \tilde{M}_\sigma(\zeta_1, \zeta_2) \\ &= \frac{e^{-\frac{1}{2}(\xi_1 + \xi_2) - \frac{1}{2}(\eta_1^2 + \eta_2^2)}}{\sqrt{\pi} \left(1 + \frac{i}{2}\sigma^{3/2}(6 \log \sigma)^{1/4}(\eta_1 - \eta_2)\right)} + o(1) \\ &\rightarrow M_{P2}(\zeta_1, \zeta_2) \text{ as } \sigma \rightarrow \infty. \end{aligned}$$

It is easy to verify that  $M_\sigma(\zeta_2, \zeta_1) = \overline{M_\sigma(\zeta_1, \zeta_2)}$  for all  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$ . To prove weak convergence and convergence of the last particle distributions it suffices by Lemma 2.1 to show that there is a function  $B_\infty$ , which is integrable on  $(\xi_0, \infty) \times \mathbb{R}$  for every  $\xi_0 \in \mathbb{R}$ , such that  $B_\infty(\zeta) \geq |\tilde{M}_\sigma(\zeta, \zeta)|$  for all sufficiently large  $\sigma$ . Now

$$\begin{aligned} M_\sigma(\zeta, \zeta) &\leq \frac{1}{4\pi^{5/2}} \int_\gamma \int_\gamma \left| \frac{e^{-\frac{1}{2}(\sigma v - \eta)^2 + \frac{4}{3}(v^3 + u^3) + i\xi(v+u) - \frac{1}{2}(\sigma u + \eta_1)^2}}{i(u+v)} \right| du dv \\ &= \frac{e^{\delta^2\sigma^2 + \frac{2}{3}\delta^3 - \eta^2 - 2\delta\xi}}{4\pi^{5/2}} \iint_{\mathbb{R}^2} \frac{e^{-\frac{1}{2}(\sigma^2 + 2\delta)(t^2 + s^2) + \sigma\eta(t-s)}}{|s+t+2\delta i|} dt ds \\ &\leq \frac{\exp\left\{\delta^2\sigma^2 + \frac{2}{3}\delta^3 - \frac{2\delta\eta^2}{\sigma^2 + 2\delta} - 2\delta\xi\right\}}{4\pi^{3/2}\delta(\sigma^2 + 2\delta)}, \end{aligned}$$

so choosing  $\delta = 1/2a_\sigma$  gives

$$\begin{aligned} \tilde{M}_\sigma(\zeta, \zeta) &\leq \frac{a_\sigma b_\sigma e^{\delta^2\sigma^2 + \frac{2}{3}\delta^3}}{4\pi^{3/2}\delta(\sigma^2 + 2\delta)} \exp\left\{-\frac{2\delta b_\sigma^2 \eta^2}{\sigma^2 + 2\delta} - 2\delta(c_\sigma + a_\sigma \xi)\right\} \\ &= \frac{\exp\left\{\frac{\sqrt{6}(\log \sigma)^{3/2}}{\sigma^3}\right\}}{\sqrt{\pi} \left(1 + \frac{\sqrt{6} \log \sigma}{\sigma^3}\right)} \exp\left\{-\frac{\eta^2}{\left(1 + \frac{\sqrt{6} \log \sigma}{\sigma^3}\right)} - \xi\right\} \\ &\leq \frac{e^{1 - \frac{1}{2}\eta^2 - \xi}}{\sqrt{\pi}} \\ (3.9) \quad &=: B_\infty(\xi, \eta) \end{aligned}$$

for all sufficiently large  $\sigma$ , which concludes the proof.  $\square$

## 4. PROOF OF THEOREM 2.5

The proof is organized as follows: In Section 4.1 an integral representation formula for the sum of products of Hermite polynomials appearing in (1.5) is derived. The resulting representation of the correlation kernel and a general discussion of the saddle point arguments used to calculate its asymptotics are given in 4.2. Sections 4.4 and 4.6 contain the proofs of part (i) and (ii) of Theorem 2.5 respectively. The two cases of the proof are preceded by Sections 4.3 and 4.5 respectively, with a series of lemmas estimating the integrals appearing in the correlation kernel for the two cases.

In the asymptotic estimates we use the following notation:  $C$  and  $C_i$ ,  $i = 1, 2, \dots$  are generic positive constants, and the occurrence of the same symbol in different chains of inequalities need not denote the same number. If  $y_n > 0$  and  $x_n/y_n \rightarrow 0$  as  $n \rightarrow \infty$ , we may write  $x_n = o(y_n)$  or, provided  $x_n \geq 0$ ,  $x_n \ll y_n$ . The relation  $x_n = \mathcal{O}(y_n)$  means that there is a positive constant  $C$  such that  $|x_n| \leq Cy_n$  for every  $n$ . If there is a positive constant  $C$  such that  $x_n/C < y_n < Cx_n$  for every  $n$ , we write  $x_n \asymp y_n$ . In the proofs of lemmas and theorems where  $\zeta = (\zeta_1, \zeta_2)$  is explicitly stated to be fixed, constants may depend on  $\zeta$ .

**4.1. Integral formula for the Hermite polynomials.** The proof of Theorem 2.5 essentially reduces to calculating asymptotics of the (appropriately rescaled) correlation kernel  $K_n^{\tau_n}$ . The key to this analysis is to find a suitable representation of the Hermite polynomials; because of the scale factor  $\tau^k$  appearing in (1.5) we cannot use the Christoffel-Darboux formula to simplify the sum, which is the standard technique. The formula derived in Lemma 4.1 differs from the one used in [5] for the special case  $\sigma_n = cn^{-1/3}$ , and it enables us to carry out a more complete analysis.

**Lemma 4.1.** *Let  $r_1, r_2$  and  $\tau$  be positive real numbers such that  $r_1 < \tau r_2$  and define the contours*

$$(4.1) \quad [-\pi, \pi] \ni t \mapsto \gamma_{r_1}(t) = r_1 e^{it} \in \mathbb{C}$$

and

$$(4.2) \quad \mathbb{R} \ni t \mapsto \Gamma_{r_2}(t) = r_2 + it \in \mathbb{C}$$

in the complex plane. Then, for any positive integer  $n$  and all complex numbers  $z_1$  and  $z_2$ , the identity

$$(4.3) \quad \sum_{k=0}^{n-1} \tau^k h_k(z_1) h_k(z_2) = \frac{\tau^n e^{z_2^2}}{2\pi^2} \oint_{\gamma_{r_1}} \int_{\Gamma_{r_2}} \left( \frac{w_2}{w_1} \right)^n \frac{e^{w_2^2 - 2z_2 w_2 + 2z_1 w_1 - w_1^2}}{w_1 - \tau w_2} dw_2 dw_1$$

holds.

*Proof.* Recall that the orthonormal Hermite polynomials can be written

$$(4.4) \quad h_k(z) = \pi^{1/4} \sqrt{2^k k!} H_k(z),$$

where

$$(4.5) \quad H_k(z) = (-1)^k e^{z^2} \frac{d^k(e^{-z^2})}{dz^k},$$

and that the renormalized polynomials  $H_k$  satisfy the generating function relation

$$(4.6) \quad e^{2wz-w^2} = \sum_{n=1}^{\infty} \frac{H_n(z)w^n}{n!}$$

for all complex numbers  $z$  and  $w$ . Note that for any choice of real  $r_2$ ,

$$(4.7) \quad e^{-z^2} = \frac{1}{i\sqrt{\pi}} \int_{\Gamma_{r_2}} e^{w^2-2zw} dw.$$

Using Equations (4.5) and (4.7) gives a representation

$$(4.8) \quad H_k(z) = \frac{2^k e^{z^2}}{i\sqrt{\pi}} \int_{\Gamma_{r_2}} w^k e^{w^2-2zw} dw.$$

On the other hand, Equation (4.6) and the residue theorem yield, for any choice of  $r_1 > 0$ ,

$$(4.9) \quad \frac{1}{2\pi i} \oint_{\gamma_{r_1}} \frac{e^{2wz-w^2}}{w^{k+1}} dw = \frac{H_k(z)}{k!}.$$

Combining the representations (4.8) and (4.9) gives, for any  $\tau > 0$  such that  $|\tau r_2| > r_1$ ,

$$\begin{aligned} \sum_{k=0}^{n-1} \tau^k h_k(z_1) h_k(z_2) &= \sum_{k=0}^{n-1} \frac{H_k(z_1) H_k(z_2)}{\sqrt{\pi} k! (2/\tau)^k} \\ &= - \sum_{k=0}^{n-1} \frac{\tau^k e^{z_2^2}}{2\pi^2} \oint_{\gamma_{r_1}} \int_{\Gamma_{r_2}} \frac{w_2^k}{w_1^{k+1}} e^{2w_1 z - w_1^2 + w^2 - 2zw} dw_2 dw_1 \\ &= - \frac{e^{z_2^2}}{2\pi^2} \oint_{\gamma_{r_1}} \int_{\Gamma_{r_2}} \sum_{k=0}^{n-1} \left( \frac{\tau w_2}{w_1} \right)^k \frac{e^{2w_1 z - w_1^2 + w^2 - 2zw}}{w_1} dw_2 dw_1 \\ &= - \frac{e^{z_2^2}}{2\pi^2} \oint_{\gamma_{r_1}} \int_{\Gamma_{r_2}} \frac{\left( \frac{\tau w_2}{w_1} \right)^n - 1}{\tau w_2 - w_1} e^{2w_1 z - w_1^2 + w^2 - 2zw} dw_2 dw_1 \\ &= \frac{\tau^n e^{z_2^2}}{2\pi^2} \oint_{\gamma_{r_1}} \int_{\Gamma_{r_2}} \left( \frac{w_2}{w_1} \right)^n \frac{e^{w_2^2 - 2z_2 w_2 + 2z_1 w_1 - w_1^2}}{w_1 - \tau w_2} dw_2 dw_1 \end{aligned}$$

as claimed, where the last equality follows since

$$\oint_{\gamma_{r_1}} \frac{e^{2w_1 z - w_1^2 + w^2 - 2zw}}{\tau w_2 - w_1} dw_1 = 0,$$

by Cauchy's theorem. □

**4.2. The saddle point argument.** Identifying  $\zeta_j = \xi_j + i\eta_j \in \mathbb{C}$  with  $(\xi_j, \eta_j) \in \mathbb{R}^2$  and using Lemma 4.1, Equation (1.5) can be written

$$(4.10) \quad K_n^\tau(\zeta_1, \zeta_2) = \frac{n\tau^n}{2\pi^{5/2}\sqrt{(1-\tau^2)}} \exp \left\{ -\frac{n}{2} \left( -\frac{\bar{\zeta}_2^2}{\tau} + \frac{\xi_1^2 + \xi_2^2}{1+\tau} + \frac{\eta_1^2 + \eta_2^2}{1-\tau} \right) \right\} \\ \times \oint_{\gamma_{r_1}} \int_{\Gamma_{r_2}} \frac{e^{n \log w_2 + w_2^2 - \sqrt{\frac{2n}{\tau}} \bar{\zeta}_2 w_2 - (n \log w_1 + w_1^2 - \sqrt{\frac{2n}{\tau}} \zeta_1 w_1)}}{w_1 - \tau w_2} dw_2 dw_1,$$

for appropriate choices of  $r_1$  and  $r_2$ , whenever  $0 < \tau < 1$ . (For the Ginibre ensemble,  $\tau = 0$ , the correlation kernel has the form

$$K_n^0(\zeta_1, \zeta_2) = \frac{n}{\pi} \sum_{k=0}^{n-1} \frac{(\zeta_1 \bar{\zeta}_2)^k}{k!} e^{-\frac{n}{2}(|\zeta_1|^2 + |\zeta_2|^2)},$$

and a representation

$$\sum_{k=0}^{n-1} \frac{(\zeta_1 \bar{\zeta}_2)^k}{k!} = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{\zeta_1 \bar{\zeta}_2 w}}{w^n(1-w)} dw$$

analogous to (4.3) gives a (simpler) saddle point argument, paralleling the  $\tau > 0$  case. The details will be skipped.)

The correlation kernel of the rescaled point process  $\tilde{Z}_n^{\tau_n}$  is given by

$$(4.11) \quad \tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2) \\ = a_n b_n \frac{\tau_n}{2n} K_n^{\tau_n} \left( \sqrt{\frac{\tau_n}{2n}}(c_n + a_n \xi_1 + i b_n \eta_1), \sqrt{\frac{\tau_n}{2n}}(c_n + a_n \xi_2 + i b_n \eta_2) \right),$$

where we will see that the scaling parameters

$$(4.12) \quad (a_n, b_n, c_n) := \sqrt{\frac{2n}{\tau_n}}(\tilde{a}_n, \tilde{b}_n, \tilde{c}_n),$$

should be chosen as in the statement of the theorem.

Define the analytic function

$$(4.13) \quad f_n(w) = n \log w + w^2 - c_n w,$$

where we choose the principal branch of the logarithm. We expect  $\tilde{c}_n$  to be close to  $(1 + \tau_n)$ , the rightmost edge of the spectrum on the global scale, so the main contribution to the exponent in the integral in (4.10) should be  $f_n(w_2) - f_n(w_1)$ . The idea of the proof is now to calculate the large  $n$  asymptotics of  $\tilde{K}_n^{\tau_n}$  by a saddle point argument for  $f_n$ . Define the shift parameter

$$(4.14) \quad \delta_n := c_n - \sqrt{\frac{2n}{\tau_n}}(1 + \tau_n).$$



Provided  $0 \leq x_n = o(1)$ , where

$$(4.15) \quad x_n = \frac{\delta_n}{\sqrt{n}(1-\tau_n)^2},$$

which will turn out always to be the case,  $f_n$  has two distinct, real, positive saddle points,  $w_+ > w_-$ , solving the equation

$$(4.16) \quad f'_n(w) = \frac{n}{w} + 2w - c_n = 0.$$

Explicitly,

$$(4.17) \quad w_+ = \frac{c_n}{4} \left( 1 + \sqrt{1 - \frac{8n}{c_n^2}} \right) = \sqrt{\frac{n}{2\tau_n}} + \frac{\delta_n}{2(1-\tau_n)} + \mathcal{O}\left(\frac{\delta_n^2}{\sqrt{n}(1-\tau_n)^3}\right),$$

and

$$(4.18) \quad w_- = \frac{c_n}{4} \left( 1 - \sqrt{1 - \frac{8n}{c_n^2}} \right) = \sqrt{\frac{\tau_n n}{2}} - \frac{\tau_n \delta_n}{2(1-\tau_n)} + \mathcal{O}\left(\frac{\delta_n^2}{\sqrt{n}(1-\tau_n)^3}\right),$$

where the asymptotics are given by an expansion of the square root,

$$(4.19) \quad \sqrt{1 - \frac{8n}{c_n^2}} = \frac{(1-\tau_n)}{(1+\tau_n)} \left( 1 + \frac{(2\tau_n)^{3/2}}{(1+\tau_n)} x_n - \frac{\tau_n^2(3-2\tau_n+3\tau_n^2)}{(1+\tau_n)^2} x_n^2 + \mathcal{O}(x_n^3) \right).$$

To analyze the behaviour of  $f_n$  close to the saddle points we observe that

$$(4.20) \quad f''_n(w_+) = 2 - \frac{n}{w_+^2} = 2(1-\tau_n) + \mathcal{O}\left(\frac{\delta_n}{\sqrt{n}(1-\tau_n)}\right) > 0,$$

$$(4.21) \quad f''_n(w_-) = -\frac{2}{\tau_n}(1-\tau_n) + \mathcal{O}\left(\frac{\delta_n}{\sqrt{n}(1-\tau_n)}\right) < 0,$$

and

$$f_n^{(k)}(w) = (-1)^{k-1} (k-1)! n w^{-k}, \text{ for } k \geq 3.$$

For sequences  $(\tau_n)_{n=1}^\infty$  such that  $|f''_n(w_-)|$  and  $|f''_n(w_+)|$  become very small, we cannot ignore the third order terms in the Taylor expansions of  $f_n$  at the saddle points; more specifically this happens whenever

$$\lim_{n \rightarrow \infty} \left| \frac{f_n'''(w_+)^2}{f_n''(w_+)^3} \right| > 0,$$

and similarly at  $w_-$ , that is, whenever  $\sigma_n = \mathcal{O}(1)$ . This results in completely different asymptotic behaviour of the integral in (4.10) depending on whether or not  $\sigma_n = \mathcal{O}(1)$ , and the choices of contours and parameters in the general parameterization of the integral, given in Lemma 4.2, will differ in the two cases.

In principle, in view of (4.20) and (4.21), we would like to choose  $r_1 = w_-$  and  $r_2 = w_+$  in (4.10) in order to pick up a Gaussian integral at each of the saddle points. However, since

$$(4.22) \quad \tau_n w_+ - w_- = \frac{\tau_n \delta_n}{(1-\tau_n)} (1 + \mathcal{O}(x_n)),$$

the integral in (4.10) will not converge for this choice of  $r_1$  and  $r_2$  unless  $\delta_n > 0$ , so if  $\delta_n = 0$  a slight modification will be necessary.

For  $r_1$  and  $r_2$  (depending on  $n$ ) to be specified, parameterize the contours of integration

$$(4.23) \quad \begin{cases} \gamma_{r_1} : s \mapsto r_1 e^{i\theta_n s}, |s| \leq \pi/\theta \\ \Gamma_{r_2} : t \mapsto r_2 + i\alpha_n t, t \in \mathbb{R}, \end{cases}$$

where  $\theta_n$  and  $\alpha_n$  are positive parameters. Moving out the main contributing factor from the integral, we get the following representation of the correlation kernel:

**Lemma 4.2.** *Let  $r_1$  and  $r_2$  be any positive real numbers such that  $\tau_n r_2 - r_1 > 0$ . For  $t, v \in \mathbb{R}$ , define*

$$(4.24) \quad \begin{aligned} \Xi_{r_2}^{\zeta_2}(t) &:= f_n(r_2 + i\alpha_n t) - f_n(r_2) - i\alpha_n t(a_n \xi_2 - ib_n \eta_2) \\ &= n \log \left( 1 + \frac{i\alpha_n t}{r_2} \right) - \alpha_n^2 t^2 - i\alpha_n t(c_n + a_n \xi_2 - ib_n \eta_2 - 2r_2) \end{aligned}$$

and

$$(4.25) \quad \begin{aligned} \Phi_{r_1}^{\zeta_1}(v) + i\Psi_{r_1}^{\zeta_1}(v) &:= -f_n(r_1 e^{iv}) + f_n(r_1) + r_1(e^{iv} - 1)(a_n \xi_1 + ib_n \eta_1) \\ &= r_1^2(1 - 2\cos 2v) + r_1 c'_n(\cos v - 1) - r_1 b_n \eta_1 \sin v \\ &\quad + i((-n+1)v - r_1^2 \sin 2v + r_1 c'_n \sin v - r_1 b_n \eta_1(\cos v - 1)) \end{aligned}$$

where  $c'_n = c_n + a_n \xi_1$  and  $\Phi_{r_1}^{\zeta_1}$  and  $\Psi_{r_1}^{\zeta_1}$  are the real and imaginary parts of the right hand side of (4.25) respectively.

The correlation kernel  $\tilde{K}_n^{\tau_n}$  of the rescaled determinantal point process  $\tilde{Z}_n^{\tau_n}$  can then be expressed

$$(4.26) \quad \begin{aligned} \tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2) &= \frac{-r_1 \theta_n \alpha_n a_n b_n \tau_n^{n+1}}{4\pi^{5/2} \sqrt{(1 - \tau_n^2)}} \exp \left\{ \frac{(c_n + a_n \xi_2)^2 - \tau_n(c_n + a_n \xi_1)^2}{4(1 + \tau_n)} \right. \\ &\quad \left. - \frac{b_n^2(\tau_n \eta_1^2 + \eta_2^2)}{4(1 - \tau_n)} - \frac{ib_n \eta_2(c_n + a_n \xi_2)}{2} \right\} \\ &\quad \times \exp \{ f_n(r_2) - f_n(r_1) + a_n(r_1 \xi_1 - r_2 \xi_2) + ib_n(r_1 \eta_1 + r_2 \eta_2) \} \\ &\quad \times \iint_{\{|s| < \pi/\theta\} \times \mathbb{R}} F_{r_1, r_2}^{\zeta_1, \zeta_2}(s, t) dt ds, \end{aligned}$$

where

$$(4.27) \quad F_{r_1, r_2}^{\zeta_1, \zeta_2}(s, t) := \frac{\exp \{ \Phi_{r_1}^{\zeta_1}(\theta_n s) + i\Psi_{r_1}^{\zeta_1}(\theta_n s) + \Xi_{r_2}^{\zeta_2}(t) \}}{r_1 e^{i\theta_n s} - \tau_n(r_2 + i\alpha_n t)}.$$

The technical part of the proof of Theorem 2.5 has now been reduced to estimating the integral

$$(4.28) \quad G(\zeta_1, \zeta_2) := \iint_{\{|s| < \pi/\theta\} \times \mathbb{R}} F_{r_1, r_2}^{\zeta_1, \zeta_2}(s, t) dt ds.$$

In order to estimate the  $s$ -integral, we need to establish that the real function

$$\Phi_{r_1}^\zeta(v) = r_1^2(1 - 2\cos 2v) + r_1 c'_n(\cos v - 1) - r_1 b_n \eta_1 \sin v$$

decreases monotonically as one moves away from its global maximum. This is the content of the following lemma.

**Lemma 4.3.** *Let  $\zeta \in \mathbb{R}^2$  be fixed. Put  $v_0 = -b_n \eta / (c'_n - 4r_1)$  and choose  $r_1 = w_-$  if  $\sigma_n \rightarrow \infty$ , and  $r_1 = w_-(1 - \epsilon_n n^{-1/3})$  otherwise, where  $n^{-1/3} \ll \epsilon_n = o(1)$ . Then for every sufficiently large  $n$  there is a  $\rho_n$  with*

$$\begin{cases} \rho_n < C_1 \frac{1}{\epsilon_n n^{1/3}} \text{ if } \sigma_n = o(1) \\ \rho_n < C_2 \frac{b_n^2}{n(1-\tau_n)^3} \text{ otherwise,} \end{cases}$$

such that if  $v_1 \in (-\pi/2, \pi/2) \setminus (v_0 - \rho_n, v_0 + \rho_n)$ ,  $v_2 \in [-\pi, \pi]$  and

$$|v_2 - v_0| > |v_1 - v_0|,$$

then

$$\Phi_{r_1}^\zeta(v_1) > \Phi_{r_1}^\zeta(v_2).$$

*Proof.*  $\Phi_{r_1}^\zeta(v)$  is differentiable everywhere, so any local extreme points  $v$  satisfy  $\Phi_{r_1}^{\zeta'}(v) = 0$ , or

$$(4.29) \quad \sin v \left( 4r_1 - \frac{c'_n}{\cos v} \right) - b_n \eta = 0,$$

provided  $\eta \neq 0$ . Put  $x = \sqrt{\frac{\tau_n n}{2}} - r_1$ . Now

$$(4.30) \quad r_1(c'_n - 4r_1) = n(1 - \tau_n) + (\delta_n + 2x(3 - 1/\tau_n))\sqrt{\frac{\tau_n n}{2}} + \mathcal{O}(x^2) + \mathcal{O}(a_n \sqrt{n}) > 0$$

if  $n$  is sufficiently large, so

$$\left| 4r_1 - \frac{c'_n}{\cos v} \right| \geq c'_n - 4r_1 = \mathcal{O}(x) \text{ if } \sigma_n \rightarrow 0,$$

and

$$\left| 4r_1 - \frac{c'_n}{\cos v} \right| \geq c'_n - 4r_1 = \mathcal{O}(\sqrt{n}(1 - \tau_n)) \text{ otherwise.}$$

In both cases,

$$\left| \frac{b_n}{4r_1 - c'_n / \cos v} \right| = o(1).$$

Therefore, for any solution  $v$  to (4.29),  $\sin v = \mathcal{O}(v_0) = o(1)$ . Consider first the case that  $|v| = o(1)$ . Then there are numbers  $\epsilon_1$  and  $\epsilon_2$  with  $|\epsilon_1| < |v^3|/6$  and  $|\epsilon_2| < v^2$  such that (4.29) can be written

$$(v - \epsilon_1)(4r_1 - c'_n(1 + \epsilon_2)) - b_n \eta = 0,$$

which implies that there is a constant  $C$  such that

$$v \in (v_0 - \rho_n, v_0 + \rho_n),$$

where

$$\rho_n < C' \frac{v_0 c'_n \epsilon_2}{c'_n - 4r_1} < C \frac{v_0 c'_n b_n^2 \eta^2}{(c'_n - 4r_1^2)^3}.$$

Similarly, if  $|v - \pi| = o(1)$  a Taylor expansion of (4.29) gives the condition  $v \in (\tilde{v}_0 - \rho_n, \tilde{v}_0 + \rho_n)$  where  $\tilde{v}_0 = \pi - b_n \eta / (c'_n + 4r_1)$ . It follows that  $\Phi_{r_1}^\zeta(v)$  is strictly decreasing on  $[v_0 + \rho_n, \tilde{v}_0 - \rho_n]$  and strictly increasing on  $[\tilde{v}_0 - 2\pi + \rho_n, v_0 - \rho_n]$ . Clearly, any possible local maxima of  $\Phi_{r_1}^\zeta(v)$  near  $\tilde{v}_0$  are smaller than, say,  $\Phi_{r_1}^\zeta(\pm\pi/2)$ , so evaluating the order of magnitude of  $\rho_n$  and remembering that  $\Phi_{r_1}^\zeta$  is  $2\pi$ -periodic gives the conclusion.  $\square$

**4.3. Estimates on  $G(\zeta_1, \zeta_2)$  when  $\sigma_n$  tends to infinity.** The two cases, depending on whether or not  $\sigma_n = \mathcal{O}(1)$ , require different choices of the parameters. In this section, estimates on  $G(\zeta_1, \zeta_2)$  are provided for the case  $\sigma_n \rightarrow \infty$ . In this regime, the contribution to the exponent in  $F_{r_1, r_2}^{\zeta_1, \zeta_2}$  from the smaller terms depending on the  $\eta$ -variables is not negligible, so the main contribution to  $G(\zeta_1, \zeta_2)$  actually comes from intervals  $I_n, J_n$  which in general do not contain the saddle points of  $f_n$ .

First, the main contribution to the integral is estimated in Lemma 4.4, then it is shown in Lemma 4.5 that the remaining contribution is negligible, and finally in Lemma 4.6 the main contribution is explicitly evaluated. These estimates provide the basis for the proof of part (i) of Theorem 2.5 in Section 4.4.

Throughout this section we will suppose that  $\sigma_n \rightarrow \infty$  as  $n \rightarrow \infty$  and that

$$(4.31) \quad \delta_n \asymp \sqrt{(1 - \tau_n) \log \sigma_n} = n^{-2/3} \sigma_n \sqrt{\log \sigma_n},$$

$$(4.32) \quad a_n \asymp \sqrt{\frac{(1 - \tau_n)}{\log \sigma_n}} = \frac{n^{-2/3} \sigma_n}{\sqrt{\log \sigma_n}},$$

and

$$(4.33) \quad b_n \asymp \left( \frac{n(1 - \tau_n)^5}{\log \sigma_n} \right)^{1/4} = \frac{n^{-2/3} \sigma_n^{5/2}}{(\log \sigma_n)^{1/4}}.$$

Since  $\delta_n > 0$ , we may choose  $r_1 = w_-$  and  $r_2 = w_+$  by (4.22). Now choose the parameters  $\alpha_n$  and  $\theta_n$  so that the exponent in  $F_{r_1, r_2}^{\zeta_1, \zeta_2}(s, t)$  becomes of order  $-s^2/2 + t^2/2$ , namely

$$(4.34) \quad \alpha_n = \sqrt{\frac{w_+}{2(w_+ - w_-)}} = \frac{1}{\sqrt{2(1 - \tau_n)}} \left( 1 - \frac{\tau_n^{3/2} \delta_n}{\sqrt{2n}(1 - \tau_n)^2} + \mathcal{O}(x_n^2) \right),$$

and

$$(4.35) \quad \theta_n = \frac{1}{\sqrt{2w_-(w_+ - w_-)}} = \frac{1}{\sqrt{n(1 - \tau_n)}} \left( 1 - \frac{\tau_n^{3/2} \delta_n}{\sqrt{2n}(1 - \tau_n)^2} + \mathcal{O}(x_n^2) \right).$$

We observe that by definition the parameters are related through the equation

$$(4.36) \quad \alpha_n^2 = 1/2 + \theta_n^2 w_-^2$$

and that

$$(4.37) \quad \tau_n \alpha_n^2 - \theta_n^2 w_-^2 = \frac{\tau_n^{3/2} \delta_n}{\sqrt{2n}(1 - \tau_n)^2} (1 + \mathcal{O}(x_n)).$$

First we estimate the contribution to  $G(\zeta_1, \zeta_2)$  from near the (slightly shifted) saddle points.

**Lemma 4.4.** *Put  $s_0 = -\theta_n w_- b_n \eta_1$  and  $t_0 = -\alpha_n b_n \eta_2$  and let  $M_n$ , such that  $1 \ll M_n \ll \sqrt{\log \sigma_n}$ , be given. Define the sets  $I_n = \{s : |s - s_0| \leq M_n\}$  and  $J_n = \{t : |t - t_0| \leq M_n\}$  and let  $\phi_n = nb_n^3(w_+ - w_-)^{-3}/24$ . Then, for every fixed  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$ ,*

$$\begin{aligned} & \iint_{I_n \times J_n} F_{w_-, w_+}^{\zeta_1, \zeta_2}(s, t) dt ds \\ &= e^{i\phi_n(\eta_2^3 - \eta_1^3)} \iint_{I_n \times J_n} \frac{e^{-\frac{1}{2}t^2 - \frac{1}{2}s^2 - i\alpha_n t(a_n \xi_2 - ib_n \eta_2) + iw_- \theta_n s(a_n \xi_1 + ib_n \eta_1)}}{w_- - \tau_n w_+ + iw_- \theta_n s - i\tau_n \alpha_n t} dt ds \\ & \quad + o(1) \frac{e^{\frac{1}{2}(t_0^2 + s_0^2)}}{|w_- - \tau_n w_+ + iw_- \theta_n s_0 - i\tau_n \alpha_n t_0|}. \end{aligned}$$

*Proof.* We expand the exponent in  $F_{w_-, w_+}^{\zeta_1, \zeta_2}(s, t)$  and estimate for  $s \in I_n$  and  $t \in J_n$ . Note that, since

$$w_-(c'_n - w_-)\theta_n^2 = 1 + \frac{a_n \xi_1}{(c_n - 4w_-)} = 1 + \mathcal{O}\left(\frac{1}{\sqrt{(1 - \tau_n)n \log \sigma_n}}\right),$$

the expansion of the exponent becomes

$$\begin{aligned} & \Xi_{w_+}^{\zeta_2}(t) + \Phi_{w_-}^{\zeta_1}(\theta_n s) + i\Psi_{w_-}^{\zeta_1}(\theta_n s) \\ &= -\frac{1}{2}t^2 - i\alpha_n t(a_n \xi_2 - ib_n \eta_2) - \frac{1}{2}s^2 + iw_- \theta_n s(a_n \xi_1 + ib_n \eta_1) \\ & \quad - i\frac{n\alpha_n^3}{3w_+^3}t^3 + i\frac{\theta_n^3 w_-(8w_- - c'_n)}{6}s^3 - i\frac{w_- \theta_n^2 b_n \eta_1}{2}s^2 \\ & \quad + \mathcal{O}\left(\frac{b_n^4}{n(1 - \tau_n)^4}\right) + \mathcal{O}\left(\frac{(1 - \tau_n)}{\log \sigma_n}\right). \end{aligned} \quad (4.38)$$

The imaginary second and third order terms in (4.38) are in general not small in  $I_n \times J_n$ , however they are almost constant. For the term involving  $\eta_2$ ,

$$(4.39) \quad \left| i\phi_n \eta_2^3 - \left(-\frac{in\alpha_n^3 t^3}{3w_+^3}\right) \right| \leq \frac{n\alpha_n^3}{w_+^3} M_n \max(t_0^2, M_n^2) \leq C \frac{b_n^2 M_n}{\sqrt{n}(1 - \tau_n)^{5/2}}.$$

Since

$$w_-(8w_- - c'_n) = n(3\tau_n - 1) + \mathcal{O}\left(\frac{\sqrt{n}\delta_n}{1 - \tau_n}\right)$$

and

$$\frac{n\theta_n^6 w_-^3 b_n^3}{3} = \phi_n (1 + \mathcal{O}(x_n)),$$

the terms involving  $\eta_1$  can be similarly estimated;

$$\begin{aligned}
 (4.40) \quad & \left| i\phi_n \eta_1^3 - \left( i \frac{w_- \theta_n^2 b_n \eta_1}{2} s^2 - i \frac{\theta_n^3 w_- (8w_- - c'_n)}{6} s^3 \right) \right| \\
 &= \left| i\phi_n \eta_1^3 - in\theta_n^6 w_-^3 b_n^3 \eta_1^3 \left( \frac{(s/s_0)^2}{2n(\theta_n)^2} + \frac{(s/s_0)^3 (n(3\tau_n - 1)(1 + o(1)))}{6n} \right) \right| \\
 &\leq C \frac{b_n^2 M_n}{\sqrt{n}(1 - \tau_n)^{5/2}}.
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 (4.41) \quad & w_- e^{i\theta_n s} - \tau_n(w_+ + i\alpha_n t) \\
 &= (w_- + i\theta_n w_- s - \tau_n(w_+ + i\alpha_n t)) \left( 1 + \mathcal{O} \left( \frac{b_n^2}{\sqrt{n}(1 - \tau_n)\delta_n} \right) \right).
 \end{aligned}$$

Using Equations (4.38), (4.39), (4.40) and (4.41) and noting that the error terms they give rise to are all small, gives an estimate

$$\begin{aligned}
 & \left| e^{-i\phi_n(\eta_2^3 - \eta_1^3)} \iint_{I_n \times J_n} F_{w_-, w_+}^{\zeta_1, \zeta_2}(s, t) dt ds \right. \\
 & \quad \left. - \iint_{I_n \times J_n} \frac{e^{-\frac{1}{2}t^2 - \frac{1}{2}s^2 - i\alpha_n t(a_n \xi_2 - ib_n \eta_2) + iw_- \theta_n s(a_n \xi_1 + ib_n \eta_1)}}{w_- - \tau_n w_+ + iw_- \theta_n s - i\tau_n \alpha_n t} dt ds \right| \\
 & \leq |R_n| \iint_{I_n \times J_n} \left| \frac{e^{-\frac{1}{2}t^2 - \frac{1}{2}s^2 - i\alpha_n t(a_n \xi_2 - ib_n \eta_2) + iw_- \theta_n s(a_n \xi_1 + ib_n \eta_1)}}{w_- - \tau_n w_+ + iw_- \theta_n s - i\tau_n \alpha_n t} \right| dt ds \\
 & = (1 + o(1)) |R_n| \frac{e^{\frac{1}{2}(t_0^2 + s_0^2)}}{|w_- - \tau_n w_+ + iw_- \theta_n s_0 - i\tau_n \alpha_n t_0|},
 \end{aligned}$$

where  $R_n = o(1)$  and the conclusion follows.  $\square$

Next, it is shown that the contributions from outside the set  $I_n \times J_n$  to  $G(\zeta_1, \zeta_2)$  and its approximation, given in the previous lemma, are negligible.

**Lemma 4.5.** *Define  $s_0, t_0, M_n, I_n$ , and  $J_n$  as in Lemma 4.4 and let  $\theta_n^{-1}\mathbb{T}$  denote the set  $(-\pi/\theta_n, \pi/\theta_n)$ . Then, for  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$  fixed,*

$$\begin{aligned}
 (4.42) \quad & \left| \iint_{(\theta_n^{-1}\mathbb{T} \times \mathbb{R}) \setminus (I_n \times J_n)} F_{w_-, w_+}^{\zeta_1, \zeta_2}(s, t) dt ds \right| \\
 &= o(1) \frac{e^{\frac{1}{2}(t_0^2 + s_0^2)}}{|w_- - \tau_n w_+ + iw_- \theta_n s_0 - i\tau_n \alpha_n t_0|}
 \end{aligned}$$

and

$$\begin{aligned} & \left| \iint_{\mathbb{R}^2 \setminus (I_n \times J_n)} \frac{e^{-\frac{1}{2}t^2 - \frac{1}{2}s^2 - i\alpha_n t(a_n \xi_2 - ib_n \eta_2) + iw_- \theta_n s(a_n \xi_1 + ib_n \eta_1)}}{w_- - \tau_n w_+ + iw_- \theta_n s - i\tau_n \alpha_n t} dt ds \right| \\ &= o(1) \frac{e^{\frac{1}{2}(t_0^2 + s_0^2)}}{|w_- - \tau_n w_+ + iw_- \theta_n s_0 - i\tau_n \alpha_n t_0|}. \end{aligned}$$

*Proof.* We prove the estimate (4.42); the second assertion follows by the same argument. Since

$$(\theta_n^{-1}\mathbb{T} \times \mathbb{R}) \setminus (I_n \times J_n) = A_1 \cup A_2 \cup A_3,$$

where  $A_1 = I_n \times (\mathbb{R} \setminus J_n)$ ,  $A_2 = ((\theta_n^{-1}\mathbb{T}) \setminus I_n) \times J_n$  and  $A_3 = ((\theta_n^{-1}\mathbb{T}) \setminus I_n) \times (\mathbb{R} \setminus J_n)$ , we consider first the integral

$$\begin{aligned} (4.43) \quad I_{J_n^c}(s) &:= \left| \int_{\mathbb{R} \setminus J_n} \frac{e^{\Xi_{w_+}^{\zeta_2}(t)}}{w_- e^{i\theta_n s} - \tau_n(w_+ + i\alpha_n t)} dt \right| \\ &\leq \int_{\mathbb{R} \setminus J_n} \frac{\exp \left\{ n \log \left| 1 + \frac{i\alpha_n t}{w_+} \right| - \alpha_n^2 t^2 - \alpha_n b_n t \eta_2 \right\}}{|w_- + iw_- \theta_n s - \tau_n(w_+ + i\alpha_n t)|} dt \\ &\leq e^{t_0^2/2} \int_{|u| > M_n} \frac{e^{-u^2/2}}{|w_- + iw_- \theta_n s - \tau_n(w_+ + i\alpha_n(u + t_0))|} du, \end{aligned}$$

in order to estimate the integral of  $F_{w_-, w_+}^{\zeta_1, \zeta_2}$  over  $A_1$ . Fix  $s \in \mathbb{R}$  and suppose first that  $\eta_2 \neq 0$  and that

$$|iw_- \theta_n s - i\tau_n \alpha_n t_0| \asymp \frac{b_n}{(1 - \tau_n)};$$

this will generically be the case. Note that

$$\frac{b_n}{(1 - \tau_n)} \gg \tau_n w_+ - w_- \asymp \sqrt{\frac{\log \sigma_n}{(1 - \tau_n)}} \gg \alpha_n M_n.$$

We can therefore choose  $N_n$  such that

$$\alpha_n M_n \ll \alpha_n N_n \ll |w_- \theta_n s - \tau_n \alpha_n t_0| \ll \alpha_n N_n^2,$$

for example

$$N_n = \frac{\sigma_n}{(\log \sigma_n)^{1/6}}$$

will do. To see that the contribution to  $I_{J_n^c}(s)$  from  $|u| > N_n$  is negligible, note that

$$\begin{aligned}
& \int_{\pi/\theta_n > |u| > N_n} \frac{e^{-u^2/2}}{|w_- + iw_- \theta_n s - \tau_n(w_+ + i\alpha_n(u + t_0))|} du \\
& \quad \times \left( \int_{N_n > |u| > M_n} \frac{e^{-u^2/2}}{|w_- + iw_- \theta_n s - \tau_n(w_+ + i\alpha_n(u + t_0))|} du \right)^{-1} \\
& \leq \frac{\sup_{N_n > |u| > M_n} \{|w_- + iw_- \theta_n s - \tau_n(w_+ + i\alpha_n(u + t_0))|\}}{\tau_n w_+ - w_-} \\
& \quad \times \frac{\int_{N_n < |u| < L_n} e^{-u^2/2} du}{\int_{M_n < |u| < N_n} e^{-u^2/2} du} \\
& \leq C_1 \frac{\alpha_n N_n^2 \sqrt{(1 - \tau_n)}}{\sqrt{\log \sigma_n}} \frac{M_n}{N_n} e^{-N_n^2/3} \leq C N_n e^{-N_n^2/3}.
\end{aligned}$$

Equation (4.43) therefore becomes

$$\begin{aligned}
I_{J_n^c}(s) & \leq C_1 e^{t_0^2/2} \int_{N_n > |u| > M_n} \frac{e^{-u^2/2}}{|w_- + iw_- \theta_n s - \tau_n(w_+ + i\alpha_n(u + t_0))|} du \\
& \leq C \frac{e^{-M_n^2/2}}{M_n} \frac{e^{t_0^2/2}}{|w_- + iw_- \theta_n s - \tau_n(w_+ + i\alpha_n t_0)|}.
\end{aligned}$$

The (simpler) cases when  $\eta_2 = 0$  or  $|iw_- \theta_n s - i\tau_n \alpha_n t_0| \ll b_n(1 - \tau_n)^{-1}$  can be handled similarly, and lead to the same estimate. Essentially the same argument as in the proof of Lemma 4.4 now provides an estimate

$$\begin{aligned}
(4.44) \quad & \left| \iint_{A_1} F_{w, w_+}^{\zeta_1, \zeta_2}(s, t) dt ds \right| = \left| \int_{I_n} I_{J_n^c}(s) e^{\Phi_{w_-}^{\zeta_1}(\theta_n s) + i\Psi_{w_-}^{\zeta_1}(\theta_n s)} ds \right| \\
& \leq C \frac{e^{-M_n^2/2}}{M_n} e^{t_0^2/2} \int_{I_n} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n s)}}{|w_- - \tau_n w_+ + iw_- \theta_n s - i\tau_n \alpha_n t_0|} ds \\
& \leq C \frac{e^{-M_n^2/2}}{M_n} \frac{e^{t_0^2 + s_0^2/2}}{|w_- - \tau_n w_+ + iw_- \theta_n s_0 - i\tau_n \alpha_n t_0|}.
\end{aligned}$$

Now we turn to the integral over  $A_2$ . Consider, analogously with the previous estimate, the case that  $\eta_1 \neq 0$  and

$$|iw_- \theta_n s_0 - i\tau_n \alpha_n t| \gg \tau_n w_+ - w_-.$$

Choose  $N_n$  as before and let  $L_n = (n(1 - \tau_n))^{1/5}$ , so that  $L_n \gg N_n$ , but  $\theta_n L_n \ll 1$ . Then for  $t$  fixed, we proceed to show that the main contribution to

$$(4.45) \quad I_{I_n^c}(t) := \left| \int_{(\theta_n^{-1}\mathbb{T}) \setminus I_n} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n s) + i\Psi_{w_-}^{\zeta_1}(\theta_n s)}}{w_- e^{i\theta_n s} - \tau_n(w_+ + i\alpha_n t)} ds \right|$$



comes from the intervals  $M_n \leq |u| \leq N_n$ . First, by Lemma 4.3,

$$\begin{aligned}
& \int_{L_n < |u| < \frac{\pi}{\theta_n}} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n(u+s_0))}}{|w_- e^{i\theta_n(u+s_0)} - \tau_n(w_+ + i\alpha_n t)|} du \\
& \quad \times \left( \int_{M_n < |u| < L_n} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n(u+s_0))}}{|w_- e^{i\theta_n(u+s_0)} - \tau_n(w_+ + i\alpha_n t)|} du \right)^{-1} \\
& \leq \frac{\sup_{L_n > |u| > M_n} \{|w_- + iw_- \theta_n(u+s_0) - \tau_n(w_+ + i\alpha_n t)|\} e^{\Phi_{w_-}^{\zeta_1}(L_n)}}{\int_{M_n < |u| < L_n} e^{-u^2/2} du} \\
& \quad \times \int_{L_n < |u| < \frac{\pi}{\theta_n}} \frac{du}{\tau_n w_+ - w_- - w_- \cos(\theta_n(u+s_0))} \\
& \leq C_1 \frac{w_- L_n e^{\frac{1}{2}s_0^2 - \frac{1}{2}L_n^2}}{\sqrt{(\tau_n^2 w_+^2 - w_-^2)}} \frac{M_n}{e^{-M_n^2/2}} \leq C(n(1-\tau_n) \log \sigma_n)^{1/4} L_n e^{-L_n^2/3} \ll 1
\end{aligned}$$

But for  $|s| < L_n$ , we also have an estimate

$$\begin{aligned}
& \int_{N_n < |u| < L_n} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n(u+s_0))}}{|w_- e^{i\theta_n(u+s_0)} - \tau_n(w_+ + i\alpha_n t)|} du \\
& \quad \times \left( \int_{M_n < |u| < N_n} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n(u+s_0))}}{|w_- e^{i\theta_n(u+s_0)} - \tau_n(w_+ + i\alpha_n t)|} du \right)^{-1} \\
& \leq \frac{\sup_{M_n < |u| < N_n} \{|w_- + iw_- \theta_n(u+s_0) - \tau_n(w_+ + i\alpha_n t)|\}}{\tau_n w_+ - w_-} \\
& \quad \times \frac{\int_{N_n < |u| < L_n} e^{-u^2/2} du}{\int_{M_n < |u| < N_n} e^{-u^2/2} du} \\
& \leq C_1 \frac{\alpha_n N_n^2 \sqrt{(1-\tau_n)}}{\sqrt{\log \sigma_n}} \frac{M_n}{N_n} e^{-N_n^2/3} \leq C N_n e^{-N_n^2/3}.
\end{aligned}$$

Consequently, (4.45) becomes

$$\begin{aligned}
I_{I_n^c}(t) & \leq \int_{M_n < |u| < \frac{\pi}{\theta_n}} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n(u+s_0))}}{|w_- e^{i\theta_n(u+s_0)} - \tau_n(w_+ + i\alpha_n t)|} du \\
& \leq C_1 \int_{M_n < |u| < N_n} \frac{e^{\Phi_{w_-}^{\zeta_1}(\theta_n s)}}{|w_- e^{i\theta_n s} - \tau_n(w_+ + i\alpha_n t)|} ds \\
& \leq C \frac{e^{-M_n^2/2}}{M_n} \frac{e^{\frac{1}{2}s_0^2}}{|w_- - \tau_n w_+ + iw_- \theta_n s_0 - i\tau_n \alpha_n t|},
\end{aligned}$$

so

$$\begin{aligned}
 (4.46) \quad & \left| \iint_{A_2} F_{w, w_+}^{\zeta_1, \zeta_2}(s, t) dt ds \right| \\
 & \leq C_1 \frac{e^{-M_n^2/2}}{M_n} e^{s_0^2/2} \int_{J_n} \frac{e^{\Xi_{w_+}^{\zeta_2}(t)}}{|w_- - \tau_n w_+ + i w_- \theta_n s_0 - i \tau_n \alpha_n t|} dt \\
 & \leq C \frac{e^{-M_n^2/2}}{M_n} \frac{e^{t_0^2 + s_0^2/2}}{|w_- - \tau_n w_+ + i w_- \theta_n s_0 - i \tau_n \alpha_n t_0|}.
 \end{aligned}$$

Clearly,

$$(4.47) \quad \left| \iint_{A_3} F_{w, w_+}^{\zeta_1, \zeta_2}(s, t) dt ds \right| = o(1) \frac{e^{-M_n^2/2}}{M_n} \frac{e^{t_0^2 + s_0^2/2}}{|w_- - \tau_n w_+ + i w_- \theta_n s_0 - i \tau_n \alpha_n t_0|},$$

which together with (4.44) and (4.46) completes the proof.  $\square$

We close this section with an explicit evaluation of the main contribution to the approximating integral from Lemma 4.4.

**Lemma 4.6.** *For any fixed  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$ ,*

$$\begin{aligned}
 & \iint_{\mathbb{R}^2} \frac{\exp \left\{ -\frac{1}{2} t^2 - \frac{1}{2} s^2 - i \alpha_n t (a_n \xi_2 - i b_n \eta_2) + i w_- \theta_n s (a_n \xi_1 + i b_n \eta_1) \right\}}{w_- - \tau_n w_+ + i w_- \theta_n s - i \tau_n \alpha_n t} dt ds \\
 & = - \frac{2\pi \exp \left\{ -\frac{1}{2} \alpha_n^2 (a_n \xi_2 - i b_n \eta_2)^2 - \frac{1}{2} \theta_n^2 w_-^2 (a_n \xi_1 + i b_n \eta_1)^2 \right\}}{\tau_n w_+ - w_- + \tau \alpha_n^2 (a_n \xi_2 - i b_n \eta_2) + \theta_n^2 w_-^2 (a_n \xi_1 + i b_n \eta_1)} (1 + o(1)).
 \end{aligned}$$

*Proof.* Since  $\delta_n > 0$ , we have  $\tau_n w_+ - w_- > 0$  and the denominator can be written

$$\begin{aligned}
 & \frac{1}{w_- - \tau_n w_+ + i w_- \theta_n s - i \tau_n \alpha_n t} \\
 & = - \int_0^\infty \exp \{ -r(\tau_n w_+ - w_- - i w_- \theta_n s + i \tau_n \alpha_n t) \} dr,
 \end{aligned}$$

so, using Fubini's Theorem,

$$\begin{aligned}
 (4.48) \quad & \iint_{\mathbb{R}^2} \frac{\exp \left\{ -\frac{1}{2} t^2 - \frac{1}{2} s^2 - i \alpha_n t (a_n \xi_2 - i b_n \eta_2) + i w_- \theta_n s (a_n \xi_1 + i b_n \eta_1) \right\}}{w_- - \tau_n w_+ + i w_- \theta_n s - i \tau_n \alpha_n t} dt ds \\
 & = -2\pi \exp \left\{ -\frac{\alpha_n^2}{2} (-a_n \xi_2 + i b_n \eta_2)^2 - \frac{w_-^2 \theta_n^2}{2} (a_n \xi_1 + i b_n \eta_1)^2 \right\} \\
 & \quad \times \int_0^\infty \exp \{ -A_1 r^2 - (B_n + i C_n) r \} dr,
 \end{aligned}$$

where

$$\begin{aligned} A_n &= \tau_n^2 \alpha_n^2 + w_-^2 \theta_n^2 = \frac{\tau_n(1+\tau_n)}{2(1-\tau_n)}(1+o(1)), \\ B_n &= \tau_n w_+ - w_- + \tau_n \alpha_n^2 a_n \xi_2 + w_-^2 \theta_n^2 a_n \xi_1 = \frac{\tau_n \delta_n}{(1-\tau_n)}(1+o(1)), \text{ and} \\ C_n &= -\tau_n \alpha_n^2 b_n \eta_2 + w_-^2 \theta_n^2 b_n \eta_1. \end{aligned}$$

Putting  $\epsilon = B_n^{1/2} A_n^{-3/4}$  and integrating by parts gives an estimate

$$\begin{aligned} & \left| \int_0^\infty \exp \{ -A_n r^2 - (B_n + iC_n)r \} dr - \frac{1}{B_n + iC_n} \right| \\ &= \left| \frac{A_n}{B_n + iC_n} \int_0^\infty 2r \exp \{ -A_n r^2 - (B_n + iC_n)r \} dr \right| \\ &\leq \left| \frac{A_n}{B_n + iC_n} \right| \left( \int_0^\epsilon 2r e^{-B_n r} dr + \int_\epsilon^\infty 2r e^{-A_n r^2} dr \right) \\ &\leq \frac{1}{|B_n + iC_n|} \left( \frac{2\epsilon A_n}{B_n} + e^{-A_n \epsilon^2} \right) \\ (4.49) \quad &\leq C \frac{1}{|B_n + iC_n|} \left( \frac{(1-\tau_n)^{1/4}}{\sqrt{\delta_n}} + e^{-\sqrt{\log \sigma_n}} \right), \end{aligned}$$

which, inserted into (4.48), gives the conclusion.  $\square$

**4.4. Proof of Theorem 2.5, part (i).** To be able to apply Lemma 2.1 to  $(\tilde{Z}_n^{\tau_n})_{n=1}^\infty$ , we first calculate the point-wise limit of (a kernel equivalent to)  $\tilde{K}_n^{\tau_n}$ , using the estimates of Section 4.3, and then prove that, for any real  $\xi_0$ , there is an integrable function on  $(\xi_0, \infty) \times \mathbb{R}$  which dominates the functions  $\{\tilde{K}_n^{\tau_n}(\zeta, \zeta)\}_{n=1}^\infty$ .

*Proof of Theorem 2.5 (i).* Suppose  $\sigma_n \rightarrow \infty$  and choose  $\alpha_n$  and  $\theta_n$  as in the previous section, that is,

$$\alpha_n = \sqrt{\frac{w_+}{2(w_+ - w_-)}}$$

and

$$\theta_n = \frac{1}{\sqrt{2w_-(w_+ - w_-)}}.$$

The choice of scaling parameters is given by requiring that the rescaled kernel  $K_n^{\tau_n}$  have a non-trivial finite limit. To be able to control the error terms we assume from the outset that the orders of magnitude are correct, namely that the conditions

$$(4.50) \quad \delta_n \asymp \sqrt{(1-\tau_n) \log \sigma_n},$$

$$(4.51) \quad a_n \asymp \sqrt{\frac{(1-\tau_n)}{\log \sigma_n}}$$

and

$$(4.52) \quad b_n \asymp \left( \frac{n(1-\tau_n)^5}{\log \sigma_n} \right)^{1/4}.$$

are satisfied. Choose  $M_n$  such that  $1 \ll M_n \ll \sqrt{\log \sigma_n}$ . It then follows from the estimates of Lemmas 4.4, 4.5 and 4.6 that for any fixed  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$ ,

$$\begin{aligned} & e^{-i\phi_n(\eta_2^3 - \eta_1^3)} \iint_{(\theta_n^{-1}\mathbb{T}) \times \mathbb{R}} F_{w_-, w_+}^{\zeta_1, \zeta_2}(s, t) dt ds \\ &= - \frac{2\pi \exp \left\{ -\frac{1}{2}\alpha_n^2(a_n\xi_2 - ib_n\eta_2)^2 - \frac{1}{2}\theta_n^2 w_-^2(a_n\xi_1 + ib_n\eta_1)^2 \right\}}{\tau_n w_+ - w_- + \tau\alpha_n^2(a_n\xi_2 - ib_n\eta_2) + \theta_n^2 w_-^2(a_n\xi_1 + ib_n\eta_1)} (1 + o(1)). \end{aligned}$$

Since  $\tau_n w_+ - w_- > 0$  whenever  $\delta_n > 0$ , we may choose  $r_1 = w_-$  and  $r_2 = w_+$  in the representation (4.26) derived for the correlation kernel of  $\tilde{Z}_n^{\tau_n}$ . Recalling (4.36) and (4.37) this yields, after some simplification,

$$\begin{aligned} (4.53) \quad & \tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2) \\ &= \frac{a_n b_n \tau_n w_- \theta_n \alpha_n \exp \left\{ n \log \tau_n + \frac{c_n^2(1-\tau_n)}{4(1+\tau_n)} + f_n(w_+) - f_n(w_-) \right\}}{2\pi^{3/2} \sqrt{(1-\tau_n^2)} (\tau_n w_+ - w_- + \tau\alpha_n^2(a_n\xi_2 - ib_n\eta_2) + \theta_n^2 w_-^2(a_n\xi_1 + ib_n\eta_1))} \\ &\times \exp \left\{ -\frac{\tau_n a_n \delta_n}{(1-\tau_n^2)} (\xi_1 + \xi_2)(1 + o(1)) - \frac{\tau_n^{3/2} b_n^2 \delta_n}{2^{3/2} \sqrt{n}(1-\tau_n)^3} (\eta_1^2 + \eta_2^2)(1 + o(1)) \right\} \\ &\times \frac{F_n(\zeta_2)}{F_n(\zeta_1)} \exp \left\{ a_n^2 \left( \frac{1}{4(1+\tau_n)} - \frac{\alpha_n^2}{2} \right) (\xi_2^2 + \xi_1^2) \right\} (1 + o(1)), \end{aligned}$$

where  $F_n(\zeta) = \exp\{-ib_n w_- \eta + ib_n \theta_n^2 w_-^2 \xi \eta + i\phi_n \eta^3\}$  and the small  $o$  terms in the exponent are both  $\mathcal{O}(\delta_n n^{-1/2}(1-\tau_n)^{-1})$ . The correlation kernel of a determinantal point process is not uniquely defined; clearly all correlation functions remain unchanged if  $\tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2)$  is replaced by

$$(4.54) \quad K_n'(\zeta_1, \zeta_2) = \frac{F_n(\zeta_1)}{F_n(\zeta_2)} \tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2).$$

In order for the exponent in (4.53) to have a non-trivial finite limit depending on the variables  $\zeta_j$ , we choose  $a_n$  and  $b_n$  such that

$$(4.55) \quad \lim_{n \rightarrow \infty} \frac{\tau_n a_n \delta_n}{(1-\tau_n^2)} = \lim_{n \rightarrow \infty} \frac{\tau_n^{3/2} b_n^2 \delta_n}{2^{3/2} \sqrt{n}(1-\tau_n)^3} = \frac{1}{2}$$

as  $n \rightarrow \infty$ . To evaluate the constant exponential factor in (4.53) the precise asymptotics (4.19) are needed. Expansions of the logarithmic terms, and a considerable

amount of subsequent algebraic manipulation, give

$$\begin{aligned}
n \log \tau_n + \frac{c_n^2(1-\tau_n)}{4(1+\tau_n)} + f_n(w_+) - f_n(w_-) \\
= \frac{c_n^2(1-\tau_n)}{4(1+\tau_n)} + w_+^2 - w_-^2 - c_n(w_+ - w_-) + \frac{\sqrt{2\tau_n n} \delta_n}{(1-\tau_n)} \\
+ \delta_n^2 \left( \frac{\tau_n(-3-2\tau_n-2\tau_n^2-2\tau_n^3+\tau_n^4)}{4(1+\tau_n)(1-\tau_n)^3} \right) + \mathcal{O} \left( \frac{\delta_n^3}{\sqrt{n}(1-\tau_n)^5} \right) \\
= -\frac{\tau_n \delta_n^2}{(1-\tau_n^2)} + \mathcal{O} \left( \frac{\delta_n^3}{\sqrt{n}(1-\tau_n)^5} \right).
\end{aligned}$$

After an asymptotic expansion of the denominator of the constant factor in (4.53), Equation (4.54) thus becomes

$$\begin{aligned}
(4.56) \quad K_n'(\zeta_1, \zeta_2) \\
= (1 + o(1)) \frac{a_n b_n w_- \theta_n \alpha_n \sqrt{(1-\tau_n)} \exp \{ -\tau_n \delta_n^2 (1-\tau_n^2)^{-1} \}}{2\pi^{3/2} \sqrt{(1+\tau_n)} \delta_n \left( 1 + i \frac{b_n}{2\delta_n} (\eta_1 - \eta_2) \right)} e^{-\frac{1}{2}(\xi_1 + \xi_2 + \eta_1^2 + \eta_2^2)}.
\end{aligned}$$

The exact choice of  $\delta_n$  is now given by requiring that  $K_n'(\zeta_1, \zeta_2)$  have a finite limit, say

$$\lim_{n \rightarrow \infty} \frac{a_n b_n w_- \theta_n \alpha_n \sqrt{(1-\tau_n)} \exp \{ -\tau_n \delta_n^2 (1-\tau_n^2)^{-1} \}}{2\pi^{3/2} \sqrt{(1+\tau_n)} \delta_n} = \frac{1}{\sqrt{\pi}},$$

or equivalently, in view of (4.55),

$$(4.57) \quad \lim_{n \rightarrow \infty} \frac{\sqrt{(1+\tau_n)}}{2^{11/4} \tau_n^{5/4} \pi} n^{1/4} (1-\tau_n)^2 \delta_n^{-5/2} \exp \left\{ \frac{-\tau_n \delta_n^2}{(1-\tau_n^2)} \right\} = 1.$$

Equation (4.57) is satisfied for

$$\begin{aligned}
\delta_n = \sqrt{\frac{(1+\tau_n)}{4\tau_n}} \sqrt{(1-\tau_n) 6 \log \sigma_n} \\
- \sqrt{\frac{(1+\tau_n)}{\tau_n}} \sqrt{\frac{(1-\tau_n)}{6 \log \sigma_n}} \left( \frac{5}{4} \log(6 \log \sigma_n) + \log(2^{1/4} (1+\tau_n)^{3/4} \pi) \right),
\end{aligned}$$

so by (4.55) we may choose

$$a_n = \sqrt{\frac{(1+\tau_n)}{\tau_n}} \sqrt{\frac{(1-\tau_n)}{6 \log \sigma_n}},$$

and

$$b_n = \left( \frac{8}{\tau_n^2 (1+\tau_n)} \right)^{1/4} \left( \frac{n(1-\tau_n)^5}{6 \log \sigma_n} \right)^{1/4},$$

which are the choices in the statement of the theorem. Since these parameters satisfy the assumptions (4.50) through (4.52) and  $b_n \gg \delta_n$ ,

$$K'_n(\zeta_1, \zeta_2) = \frac{e^{-\frac{1}{2}(\xi_1 + \xi_2) - \frac{1}{2}(\eta_1^2 + \eta_2^2)}}{\sqrt{\pi} \left(1 + i \frac{b_n}{2\delta_n}(\eta_1 - \eta_2)\right)} (1 + o(1)) \rightarrow M_{P2}(\zeta_1, \zeta_2),$$

as  $n \rightarrow \infty$ .

It remains to prove that, for any given  $\xi_0 \in \mathbb{R}$ ,  $\tilde{K}_n^{\tau_n}(\zeta, \zeta)$  is dominated by an integrable function on  $(\xi_0, \infty) \times \mathbb{R}$  for every sufficiently large  $n$ . To estimate the integral

$$\iint_{(\theta_n^{-1}\mathbb{T}) \times \mathbb{R}} F_{w_-, w_+}^{\zeta, \zeta}(s, t) dt ds$$

from above, let  $\epsilon < 1$  be fixed and consider any  $n$  so large that

$$(4.58) \quad |\xi_0| < \epsilon \frac{(1 - \tau_n)(c_n - 4w_-)}{(1 + \tau_n)b_n^2 a_n} \asymp \frac{\log \sigma_n}{(1 - \tau_n)}.$$

For such a choice of  $n$ , it follows that

$$\epsilon_1 := \frac{3\alpha_n^2 a_n |\xi_0|}{w_+} < \epsilon \frac{4(1 - \tau_n)}{(1 + \tau_n)b_n^2} \asymp \sigma_n^{-3} \sqrt{\log \sigma_n},$$

$$\epsilon_2 := \frac{|\xi_0| a_n}{c_n - 4w_-} < \epsilon \frac{(1 - \tau_n)}{(1 + \tau_n)b_n^2},$$

and that

$$\frac{\tau_n a_n |\xi_0|}{(\tau_n w_+ - w_-)} < \epsilon \frac{\tau_n}{(1 + \tau_n)} \left(1 + \mathcal{O}\left(\frac{1}{\log \sigma_n}\right)\right) < \epsilon.$$

To capture some of the oscillatory terms in the exponent of  $F_{w_-, w_+}^{\zeta, \zeta}$ , we may again change contours of integration, by Cauchy's theorem. Replacing the real line by the contour  $\mathbb{R} - ia_n \xi / (2\alpha_n)$  in the  $t$ -integral, and putting  $t' = t + ia_n \xi / (2\alpha_n)$ , gives

$$(4.59) \quad \left| \iint_{(\theta_n^{-1}\mathbb{T}) \times \mathbb{R}} F_{w_-, w_+}^{\zeta, \zeta}(s, t) dt ds \right| \leq \frac{1}{(\tau_n w_+ - w_- - \tau_n a_n |\xi_0|)} \int_{(\theta_n^{-1}\mathbb{T})} e^{\Phi_{w_-}^{\zeta}(\theta_n s)} ds \left| \int_{\mathbb{R}} e^{\Xi_{w_+}^{\zeta}(t' - ia_n \xi / (2\alpha_n))} dt' \right|.$$

Now, using that  $f'_n(w_+) = 0$  and the inequality  $\log(1+x) \leq x$  gives an estimate

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \exp \left\{ \Xi_{w_+}^{\zeta} (t' - ia_n \xi / (2\alpha_n)) \right\} dt' \right| \\
& \leq \int_{\mathbb{R}} \exp \left\{ \operatorname{Re} \left( \Xi_{w_+}^{\zeta} (t' - ia_n \xi / (2\alpha_n)) \right) \right\} dt' \\
& \leq \int_{\mathbb{R}} \exp \left\{ -\frac{1}{2} t'^2 \left( 2\alpha_n^2 - \frac{n\alpha_n^2}{(w_+ + a_n \xi / 2)^2} \right) + \alpha_n b_n \eta t' \right. \\
& \quad \left. + n \log \left( 1 + \frac{a_n \xi}{2w_+} \right) + a_n \xi (w_+ - c_n / 2) - \frac{a_n^2 \xi^2}{4} \right\} \\
& \leq \int_{\mathbb{R}} \exp \left\{ -\frac{1}{2} t'^2 (1 - \epsilon_1) + \alpha_n b_n \eta t' - \frac{a_n^2 \xi^2}{4} \right\} \\
& = \sqrt{\frac{2\pi}{(1 - \epsilon_1)}} \exp \left\{ \frac{\alpha_n^2 b_n^2 \eta^2}{2(1 - \epsilon_1)} - \frac{a_n^2 \xi^2}{4} \right\}.
\end{aligned} \tag{4.60}$$

To obtain a uniform bound in  $n$  of the first integral, recall that

$$c'_n - 4w_- \geq (c_n - 4w_-) \left( 1 - \frac{|\xi_0| a_n}{c_n - 4w_-} \right)$$

where  $c'_n = c_n + a_n \xi$ . Suppose without loss of generality that  $\eta \leq 0$ . Then there is a  $v_\eta \in [0, \pi/2)$  such that  $\Phi_{w_-}^{\zeta \prime}(v_\eta) = 0$  and since

$$\begin{aligned}
\Phi_{w_-}^{\zeta \prime \prime}(v_\eta) &= 4w_-^2 \cos 2v_\eta - w_- c'_n \cos v_\eta + w_- b_n \eta \sin v_\eta \\
&< w_- (4w_- - c_n) < 0,
\end{aligned} \tag{4.61}$$

a saddle point argument gives

$$\begin{aligned}
(4.62) \quad & \int_{|\theta_n s| < \pi} e^{\Phi_{w_-}^{\zeta}(\theta_n s)} ds \\
& \leq C_1 e^{\Phi_{w_-}^{\zeta}(v_\eta)} \int_{\mathbb{R}} \exp \left\{ -\frac{1}{2} \left( \frac{c'_n - 4w_-}{c_n - 4w_-} \right) \left( s - \frac{v_\eta}{\theta_n} \right)^2 \right\} ds \leq C e^{\Phi_{w_-}^{\zeta}(v_\eta)}.
\end{aligned}$$

Consider the difference

$$\begin{aligned}
& \frac{b_n^2 \eta^2 w_-^2 \theta_n^2}{2(1 - \epsilon_2)} - \Phi_{w_-}^{\zeta}(v_\eta) \\
& = \frac{b_n^2 \eta^2 w_-^2 \theta_n^2}{2(1 - \epsilon_2)} + b_n w_- \eta \sin v_\eta - w_- c'_n (\cos v_\eta - 1) - w_-^2 (1 - \cos 2v_\eta) \\
& \geq -\frac{(1 - \epsilon_2) \sin^2 v_\eta}{2\theta_n^2} - c'_n w_- (\cos v_\eta - 1) - w_-^2 (1 - \cos 2v_\eta) =: g(v_\eta).
\end{aligned}$$

Since

$$g'(v_\eta) = c_n w_- \sin v_\eta \left( \frac{c'_n}{c_n} - (1 - \epsilon_2) \cos v_\eta \right) \geq 0,$$

$g$  is increasing on  $[0, \pi/2)$  so, noting that  $g(0) = 0$ , it follows from (4.62) that

$$\int_{|\theta_n s| < \pi} e^{\Phi_{w_-}^\zeta(\theta_n s)} ds \leq C \exp \left\{ \frac{b_n^2 \eta^2 w_-^2 \theta_n^2}{2(1 - \epsilon_2)} \right\}.$$

This, together with the estimates (4.59) and (4.60), inserted into (4.26) shows that

$$\begin{aligned} |\tilde{K}_n^{\tau_n}(\zeta, \zeta)| &\leq C_1 \exp \left\{ \left( \frac{(1 - \tau_n)c_n a_n}{2(1 + \tau_n)} - a_n(w_+ - w_-) \right) \xi - \frac{\tau_n a_n^2 \xi^2}{2(1 + \tau_n)} \right. \\ &\quad \left. - \frac{b_n^2 \eta^2}{2} \left( \frac{(1 + \tau_n)}{2(1 - \tau_n)} - \frac{\alpha_n^2}{(1 - \epsilon_1)} - \frac{w_-^2 \theta_n^2}{(1 - \epsilon_2)} \right) \right\} \\ &\leq C_{\xi_0} e^{-(1-\epsilon)\xi} \exp \left\{ -\eta^2 + \frac{b_n^2 \eta^2}{2} \left( \frac{\alpha_n^2 \epsilon_1}{(1 - \epsilon_1)} + \frac{w_-^2 \theta_n^2 \epsilon_2}{(1 - \epsilon_2)} \right) \right\} \\ (4.63) \quad &\leq C_{\xi_0} e^{-(1-\epsilon)\xi - (1-\epsilon)\eta^2}, \end{aligned}$$

where  $C_{\xi_0}$  denotes a constant depending on  $\xi_0$ . This provides an integrable bound on  $\tilde{K}_n^{\tau_n}(\zeta, \zeta)$ , so by Lemma (2.1) and the point-wise convergence of  $K'_n$  to  $M_{P2}$ ,  $\tilde{Z}_n^{\tau_n}$  converges weakly to  $Z_P$  and the last particle distribution  $F_n^{\tau_n}$  converges to  $F_G$ .  $\square$

**4.5. Estimates on  $G(\zeta_1, \zeta_2)$  when  $\sigma_n$  tends to a finite limit.** Throughout this section, we will suppose that  $\sigma_n \rightarrow \sigma \in [0, \infty)$ . Again, we prove a series of estimates on  $G(\zeta_1, \zeta_2)$  for this case. First, in Lemma 4.7, on the main contribution from close to the saddle points, and then in Lemmas 4.9 and 4.10 on the remaining, small contributions.

Fix the choices  $\delta_n = 0$ ,  $a_n = \sqrt{2\tau_n} n^{-1/6}$  and  $b_n = \sqrt{2\tau_n(1 - \tau_n)}$  of the scaling parameters. Since  $\delta_n = 0$ ,  $w_- - \tau_n w_+ = 0$  and we must choose contours slightly removed from the saddle points in order for  $G(\zeta_1, \zeta_2)$  to converge, say  $r_1 = w_- - \tau_n \alpha_n \epsilon_n$  and  $r_2 = w_+ + \alpha_n \epsilon_n$ , where  $\epsilon_n \ll 1$  will be specified. The quadratic terms in the expansions of  $f_n$  at the saddle points  $w_-$  and  $w_+$  may now be arbitrarily small (depending on how small  $\sigma_n$  becomes), so we choose  $\alpha_n = (2\tau_n)^{-1/2} n^{1/6}$  and  $\theta_n = n^{-1/3}$  so that the *third* order terms become of order one.

To simplify the calculations by using the fact that  $w_+$  is a saddle point of  $f_n$ , we note that, by definition,

$$(4.64) \quad F_{r_1, r_2}^{\zeta_1, \zeta_2}(s, t) = F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n) e^{-f_n(r_2) + f_n(w_+) + \alpha_n \epsilon_n (a_n \xi_2 - i b_n \eta_2)}.$$

We begin by approximating the main contribution, from close to the saddle points.

**Lemma 4.7.** *Given  $\epsilon_n$  and  $T_n$ , choose  $r_1 = w_- - \tau_n \alpha_n \epsilon_n$  and  $r_2 = w_+ + \alpha_n \epsilon_n$  and define the contour*

$$\begin{aligned} \tilde{\gamma}_n &: (-T_n, T_n) \rightarrow \mathbb{C}, \\ \tilde{\gamma}_n(t) &= t + i\epsilon_n. \end{aligned}$$

Let  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$  be fixed.

- (i) If  $\sigma > 0$  or  $\sigma_n = \mathcal{O}(n^{-2/15})$  choose  $T_n = n^{k_1}$  and  $\epsilon_n = n^{-k_2}$  for some  $0 < k_1 < 1/15$  and  $k_1 < k_2 < 2k_1$ .



- (ii) If  $\sigma = 0$  but  $\sigma_n n^{2/15} \rightarrow \infty$ , choose  $T_n = \sigma_n^{-m_1}$  and  $\epsilon_n = \sigma_n^{m_2}$ , for some  $0 < m_1 < 1/2$ , and  $m_1 < m_2 < 2m_1$ .

For these choices,

$$(4.65) \quad \sqrt{\frac{\tau_n}{2}} n^{1/6} e^{-\frac{1}{2}(\eta_1^2 + \eta_2^2)} \iint_{(-T_n, T_n)^2} F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n) dt ds \\ = - \int_{\tilde{\gamma}_n} \int_{\tilde{\gamma}_n} \frac{e^{-\frac{1}{2}(\sigma_n v - \eta_2)^2 + \frac{4}{3}v^3 + i\xi_2 v - \frac{1}{2}(\sigma_n u + \eta_1)^2 + \frac{4}{3}u^3 + i\xi_1 u}}{i(u+v)} du dv + o(1).$$

**Remark 4.8.** The case in which  $\sigma = 0$  but condition (ii) is not satisfied causes no problem, but for simplicity we omit the details.

*Proof.* Note that in both cases,  $1 \ll T_n \ll n^{1/15}$  and  $T_n^{-2} \ll \epsilon_n \ll T_n^{-1}$ . Put  $\tilde{t} = t - i\epsilon_n$  and  $\tilde{s} = s + i\epsilon_n$ . Using the expansion

$$\begin{aligned} \Phi_{r_1}^{\zeta_1}(v) + i\Psi_{r_1}^{\zeta_1}(v) &= v(-r_1 b_n \eta_1 + i(1 - n - 2r_1^2 + r_1 c'_n)) \\ &\quad - \frac{1}{2}(r_1 c'_n - 4r_1^2)v^2 - i\frac{r_1 b_n \eta_1 v^2}{2} \\ &\quad + i\frac{(8r_1^2 - r_1 c'_n)}{6}v^3 + \frac{r_1 b_n \eta_1}{6}v^3 + \mathcal{O}(nv^4) \end{aligned}$$

with  $r_1 = w_- - \tau_n \alpha_n \epsilon_n$ , and keeping in mind that  $\epsilon_n \gg n^{-2/15}$ , one obtains, after some messy but straight-forward calculations,

$$\left| \Phi_{r_1}^{\zeta_1}(\theta_n s) + i\Psi_{r_1}^{\zeta_1}(\theta_n s) - \left( i\tilde{s}(\xi_1 + i\sigma_n \eta_1) - \frac{\sigma_n^2}{2}\tilde{s}^2 + \frac{i}{3}\tilde{s}^3 \right) \right| \leq C \left( \epsilon_n + n^{-1/3} s^4 \right).$$

Using that  $f'_n(w_+) = 0$ , the expansion of the exponent in  $F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n)$  becomes

$$(4.66) \quad \Xi_{w_+}^{\zeta_2}(\tilde{t}) + \Phi_{r_1}^{\zeta_1}(\theta_n s) + i\Psi_{r_1}^{\zeta_1}(\theta_n s) \\ = -\frac{\sigma_n^2}{2}\tilde{t}^2 - \frac{i}{3}\tilde{t}^3 - i\tilde{t}(\xi_2 - i\sigma_n \eta_2) - \frac{\sigma_n^2}{2}\tilde{s}^2 + \frac{i}{3}\tilde{s}^3 + i\tilde{s}(\xi_1 + i\sigma_n \eta_1) \\ + n^{-1/3}\mathcal{O}(t^4 + s^4) + \mathcal{O}(\epsilon_n).$$

Note also that

$$\iint_{(-T_n, T_n)^2} \frac{ds dt}{|i(s-t) - 2\epsilon_n|} \leq CT_n \log \frac{T_n}{\epsilon_n}.$$

Consider first the case that either  $\sigma > 0$ , or  $\sigma_n = \mathcal{O}(n^{-2/15})$ . Then, by (4.26) and (4.66),

$$\begin{aligned}
& \left| \sqrt{\frac{\tau_n}{2}} n^{1/6} e^{-\frac{1}{2}(\eta_1^2 + \eta_2^2)} \iint_{(-T_n, T_n)^2} F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n) dt ds \right. \\
& \quad \left. - \iint_{(-T_n, T_n)^2} \frac{e^{-\frac{1}{2}(\sigma_n \bar{t} + \eta_2)^2 - \frac{i}{3}\bar{t}^3 - i\bar{t}\xi_2 - \frac{1}{2}(\sigma_n \bar{s} + \eta_1)^2 + \frac{i}{3}\bar{s}^3 + i\bar{s}\xi_1}}{i(s - t) - 2\epsilon_n} dt ds \right| \\
& \leq C_1 \left( n^{-1/3} T_n^4 + \epsilon_n \right) \\
& \quad \times \int_{(-T_n, T_n)^2} \left| \frac{e^{-\frac{1}{2}(\sigma_n \bar{t} + \eta_2)^2 - \frac{i}{3}\bar{t}^3 - i\bar{t}\xi_2 - \frac{1}{2}(\sigma_n \bar{s} + \eta_1)^2 + \frac{i}{3}\bar{s}^3 + i\bar{s}\xi_1}}{i(s - t) - 2\epsilon_n} \right| dt ds \\
& \leq C_2 \left( n^{-1/3} T_n^4 + \epsilon_n \right) \int_{(-T_n, T_n)^2} \frac{e^{\epsilon_n(\xi_1 - \xi_2) + \sigma_n^2 \epsilon_n^2}}{|i(s - t) - 2\epsilon_n|} dt ds \\
& \leq C_3 \log \frac{T_n}{\epsilon_n} \left( T_n^5 n^{-1/3} + T_n \epsilon_n \right) \\
(4.67) \quad & \leq C \log n \left( n^{5k_1 - 1/3} + n^{k_1 - k_2} \right)
\end{aligned}$$

Next, suppose  $n^{-2/15} \ll \sigma_n = o(1)$ . Then

$$\begin{aligned}
& \left| \sqrt{\frac{\tau_n}{2}} n^{1/6} \iint_{(-T_n, T_n)^2} F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n) ds dt \right. \\
& \quad \left. - \iint_{(-T_n, T_n)^2} \frac{e^{-\frac{i}{3}\bar{t}^3 - i\bar{t}\xi_2 + \frac{i}{3}\bar{s}^3 + i\bar{s}\xi_1}}{i(s - t) - 2\epsilon_n} dt ds \right| \\
& \leq C_1 \left( \sigma_n T_n + n^{-1/3} T_n^4 + \epsilon_n \right) \\
& \quad \times \int_{(-T_n, T_n)^2} \left| \frac{e^{-\frac{1}{2}(\sigma \bar{t} + \eta_2)^2 - \frac{i}{3}\bar{t}^3 - i\bar{t}\xi_2 - \frac{1}{2}(\sigma \bar{s} + \eta_1)^2 + \frac{i}{3}\bar{s}^3 + i\bar{s}\xi_1}}{i(s - t) - 2\epsilon_n} \right| dt ds \\
& \leq C_2 \left( \sigma_n T_n + n^{-1/3} T_n^4 + \epsilon_n \right) \int_{(-T_n, T_n)^2} \frac{e^{\epsilon_n(\xi_1 - \xi_2)}}{|i(s - t) - 2\epsilon_n|} dt ds \\
& \leq C_3 \left( \sigma_n T_n^2 + T_n^5 n^{-1/3} + T_n \epsilon_n \right) \log \frac{T_n}{\epsilon_n} \\
(4.68) \quad & \leq C \log \sigma_n \left( \sigma_n^{1-2m_1} + \sigma_n^{-5m_1} n^{-1/3} + \sigma_n^{-m_1+m_2} \right).
\end{aligned}$$

Letting  $u = s + i\epsilon_n$  and  $v = -(t - i\epsilon_n)$ , (4.67) and (4.68) give the conclusion.  $\square$

We turn to an estimate of the contribution to the integral of  $F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n)$  from outside the set  $(-T_n, T_n)^2$ .

**Lemma 4.9.** *Let  $T_n$  and  $\epsilon_n$  be as in Lemma 4.7 and put  $r_1 = w_- - \tau_n \alpha_n \epsilon_n$ . Let  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$  be fixed. Then*

$$(4.69) \quad \iint_{((\theta_n^{-1}\mathbb{T}) \times \mathbb{R}) \setminus (-T_n, T_n)^2} F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n) dt ds = n^{-1/6} o(1).$$

*Proof.* Note that

$$(4.70) \quad \left| F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n) \right| = \left| \frac{\exp \left\{ \Xi_{w_+}^{\zeta_2}(\tilde{t}) + \Phi_{r_1}^{\zeta_1}(\theta_n s) + i\Psi_{r_1}^{\zeta_1}(\theta_n s) \right\}}{(w_- - \tau_n \alpha_n \epsilon_n) e^{i\theta_n s} - \tau_n (w_+ + i\alpha_n \tilde{t})} \right| \\ \leq \frac{1}{\alpha_n \epsilon_n} e^{\operatorname{Re}(\Xi_{\epsilon_n}^{\zeta_2}(t))} e^{\Phi_{r_1}^{\zeta_1}(\theta_n s)},$$

so we estimate the integrals over  $s$  and  $t$  separately.

For  $t = \mathcal{O}(n^{1/3})$ , we can estimate the logarithm in  $\Xi_{w_+}^{\zeta_2}(\tilde{t})$  from above by its third order Taylor polynomial, giving

$$(4.71) \quad \operatorname{Re} \left( \Xi_{w_+}^{\zeta_2}(\tilde{t}) \right) = \frac{n}{2} \log \left( 1 + n^{-2/3}(\epsilon_n^2 + t^2) + 2\epsilon_n n^{-1/3} \right) \\ - \epsilon_n n^{2/3} - \frac{n^{1/3}}{2\tau_n} (t^2 - \epsilon_n^2) - \epsilon_n \xi_2 - \sigma_n t \eta_2 \\ \leq -\frac{1}{2} \left( \frac{\sigma_n t}{\sqrt{\tau_n}} + \sqrt{\tau_n} \eta_2 \right)^2 + \frac{1}{2} \tau_n \eta_2^2 - \epsilon_n t^2 + \frac{\sigma_n^2 \epsilon_n^2}{2\tau_n} - \epsilon_n \xi_2 \\ - Q_n(t),$$

where

$$Q_n(t) = -\frac{n^{-1/3} t^4}{4} \left( 1 - \frac{2t^2}{3n^{2/3}} \left( 1 + \frac{\epsilon_n^2}{t^2} + \frac{2\epsilon_n n^{1/3}}{t^2} \right)^3 \right).$$

To control  $Q_n(t)$ , we observe that if  $\epsilon_n n^{1/3} T_n^{-2} \geq 1/4$ , we can write

$$Q_n(t) \leq -\frac{n^{-1/3} t^4}{4} \left( 1 - C_1 \frac{t^2}{n^{2/3}} \frac{\epsilon_n^3 n}{t^6} \right) = -\frac{n^{-1/3} t^4}{4} + C_2 \epsilon_n^3,$$

which is either negative or of order  $o(1)$ . On the other hand, if  $\epsilon_n n^{1/3} T_n^{-2} > 1/4$  we have the estimate

$$Q_n(t) \leq -\frac{n^{-1/3} t^4}{4} \left( 1 - \frac{2t^2}{3n^{2/3}} (1 + 1/3) \right) \leq -\frac{n^{-1/3} t^4}{4} \left( 1 - \frac{8n^{2/3}}{9n^{2/3}} \right) < 0.$$

In both cases it follows from (4.71) that

$$\operatorname{Re} \left( \Xi_{w_+}^{\zeta_2}(\tilde{t}) \right) \leq -\epsilon_n t^2 + C_3$$

for  $|t| \leq n^{1/3}$ , so

$$(4.72) \quad \int_{T_n \leq |t| \leq n^{1/3}} e^{\operatorname{Re}(\Xi_{w_+}^{\zeta_2}(\tilde{t}))} dt \leq C \frac{e^{-\epsilon_n T_n^2}}{\epsilon_n T_n}.$$

We turn to an estimate for  $|t| > n^{1/3}$ . Changing variables  $u = n^{-1/3}t$  and letting  $\delta = (e - 2)/(2e) > 0$  gives an estimate

$$\begin{aligned}
& \int_{|t| > n^{1/3}} e^{\operatorname{Re}(\Xi_{w_+}^{\zeta_2}(\bar{t}))} dt \\
&= n^{1/3} \int_{|u| > 1} \exp \left\{ \frac{n}{2} \left( \log \left( 1 + u^2 + n^{-2/3} \epsilon_n^2 + 2\epsilon_n n^{-1/3} \right) \right) \right. \\
&\quad \left. - \frac{nu^2}{2\tau_n} - \sigma_n n^{1/3} u \eta_2 - \epsilon_n n^{2/3} + \frac{n^{1/3}}{2\tau_n} \epsilon_n^2 - \epsilon_n \xi_2 \right\} du \\
&\leq 2n^{1/3} \int_{u > 1} \exp \left\{ \frac{n}{2} \left( \log(1 + u^2) - (1 - \delta)u^2 + \frac{\tau_n \sigma_n^2 n^{-4/3} \eta_2^2}{\delta} \right) \right. \\
&\quad \left. - \frac{n^{2/3} \epsilon_n}{2} \left( 1 - \epsilon_n n^{-1/3} \left( \frac{2 + \tau_n}{2\tau_n} \right) \right) - \epsilon_n \xi_2 \right\} du \\
&\leq C_1 n^{1/3} \int_{u > 1} \exp \left\{ \frac{n}{2} \left( \log(2u^2) - (1 - \delta)u^2 \right) \right\} du \\
&\leq C_2 n^{1/3} 2^{n/2-1} \Gamma \left( \frac{n+1}{2} \right) \left( \frac{2}{n(1-\delta)} \right)^{\frac{n+1}{2}} \\
&\leq C_3 n^{-1/6} \left( \frac{6+e}{4+2e} \right)^{\frac{n+1}{2}} \\
(4.73) \quad &\leq C n^{-1/6} e^{-kn},
\end{aligned}$$

for some  $k > 0$ . Similarly, noting that  $\Phi_{r_1}^{\zeta_1}(\theta_n s) \leq -\epsilon_n s^2 + C_1 (n^{-1/3} s^4 + 1)$  and recalling that  $\epsilon_n > n^{-2/15}$ ,

$$\begin{aligned}
& \int_{T_n < |s| < \frac{\pi}{\theta_n}} e^{\Phi_{r_1}^{\zeta_1}(\theta_n s)} ds \leq C_1 \int_{T_n < |s| < \frac{\pi}{\theta_n}} e^{\Phi_{r_1}^{\zeta_1}(\theta_n s)} ds \\
&\leq C_1 \left( \frac{2\pi}{\theta_n} e^{\Phi_{r_1}^{\zeta_1}(\theta_n n^{1/12})} + \int_{T_n < |s| < n^{1/12}} e^{\Phi_{r_1}^{\zeta_1}(\theta_n s)} ds \right) \\
&\leq C_2 \left( \frac{e^{-\epsilon_n n^{1/6}}}{\theta_n} + \int_{T_n < |s| < n^{1/12}} e^{-\epsilon_n s^2} ds \right) \\
(4.74) \quad &\leq C \left( n^{1/3} e^{-n^{1/30}} + \frac{e^{-\epsilon_n T_n^2}}{\epsilon_n T_n} \right).
\end{aligned}$$

It is also clear from (4.72) and (4.73) that

$$\int_{\mathbb{R}} e^{\operatorname{Re}(\Xi_{w_+}^{\zeta_2}(\bar{t}))} dt \asymp \int_{\mathbb{R}} e^{-\epsilon_n t^2} dt \leq C \epsilon_n^{-1/2}$$

and similarly for the  $s$ -integral, so by (4.70),

$$\begin{aligned} & \left| \iint_{((\theta_n^{-1}\mathbb{T}) \times \mathbb{R}) \setminus (-T_n, T_n)} F_{r_1, w_+}^{\zeta_1, \zeta_2}(s, t - i\epsilon_n) dt ds \right| \\ & \leq Cn^{-1/6} \left( \epsilon_n^{-3/2} n^{1/3} e^{-n^{1/30}} + \epsilon_n^{-3/2} e^{-kn} + \frac{e^{-\epsilon_n T_n^2}}{\epsilon_n^{5/2} T_n} \right) = n^{-1/6} o(1), \end{aligned}$$

which concludes the proof.  $\square$

Finally, we show that the contribution to the limiting integral from outside  $(-T_n, T_n)^2$  is small.

**Lemma 4.10.** *Let  $\epsilon_n$  and  $T_n$  be as in Lemma (4.7). Then, for any fixed  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$ ,*

$$\left| \int_{\gamma \setminus \tilde{\gamma}_n} \int_{\gamma \setminus \tilde{\gamma}_n} \frac{e^{-\frac{1}{2}(\sigma v - \eta_2)^2 + \frac{i}{3}v^3 + i\xi_2 v - \frac{1}{2}(\sigma u + \eta_1)^2 + \frac{i}{3}u^3 + i\xi_1 u}}{i(u+v)} du dv \right| = o(1)$$

*Proof.* For any  $u = s + i\epsilon$ , we can estimate

$$\begin{aligned} & \left| \int_{\gamma \setminus \tilde{\gamma}_n} \frac{\exp \left\{ -\frac{1}{2}(\sigma v - \eta_2)^2 + \frac{i}{3}v^3 + i\xi_2 v \right\}}{i(u+v)} dv \right| \\ & = \left| \int_{|t| \geq T_n} \frac{\exp \left\{ -\frac{1}{2}(\sigma(t + i\epsilon_n) - \eta_2)^2 + \frac{i}{3}(t + i\epsilon_n)^3 + i\xi_2(t + i\epsilon_n) \right\}}{i(s + i\epsilon_n + t + i\epsilon_n)} dt \right| \\ & \leq \int_{|t| \geq T_n} \frac{\exp \left\{ -\frac{1}{2}(\sigma t - \eta_2)^2 - \epsilon_n t^2 + \frac{\epsilon_n^3}{3} + \frac{\sigma^2 \epsilon_n^2}{2} - \xi_2 \epsilon_n \right\}}{2\epsilon_n} dt \\ & \leq C_1 \frac{1}{\epsilon_n} \int_{|t| \geq T_n} e^{-\epsilon_n t^2} dt \leq C \frac{e^{-\epsilon_n T_n^2}}{\epsilon_n^2 T_n}. \end{aligned}$$

Therefore

$$\left| \int_{\gamma \setminus \tilde{\gamma}_n} \int_{\gamma} \frac{e^{-\frac{1}{2}(\sigma v - \eta_2)^2 + \frac{i}{3}v^3 + i\xi_2 v - \frac{1}{2}(\sigma u + \eta_1)^2 + \frac{i}{3}u^3 + i\xi_1 u}}{i(u+v)} du dv \right| \leq C \frac{e^{-\epsilon_n T_n^2}}{\epsilon_n^{5/2} T_n},$$

and by the symmetry of the variables  $u$  and  $v$  the conclusion follows.  $\square$

**4.6. Proof of Theorem 2.5, part (ii).** Using the estimates of Section 4.5 to prove the point-wise convergence of (a kernel equivalent to)  $\tilde{K}_n^{\tau_n}$  to  $M_\sigma$ , and then proving that  $\tilde{K}_n^{\tau_n}(\zeta, \zeta)$  is dominated by an integrable function, we can apply Lemma 2.1 in this case as well.

*Proof of Theorem 2.5 (ii).* Suppose  $\sigma_n \rightarrow \sigma \in [0, \infty)$  as  $n \rightarrow \infty$ . Let  $\delta_n = 0$ ,  $\alpha_n = (2\tau_n)^{-1/2} n^{1/6}$  and  $\theta_n = n^{-1/3}$ . In order to tidy up the calculations slightly,

we will actually prove the theorem for the choice of parameters  $\tilde{a} = \tau_n n^{-2/3}$  and  $\tilde{b} = \tau_n n^{-2/3} \sigma_n$ , differing by a factor  $\tau_n$  from the choice in the statement of the theorem; this clearly makes no difference in the limit. Thus  $a_n = \sqrt{2\tau_n} n^{-1/6}$  and  $b_n = \sqrt{2\tau_n(1-\tau_n)}$  as defined by (4.12). Given  $\epsilon_n$  and  $T_n$  as in Lemmas 4.7 and 4.9, choose  $r_1 = w_- - \tau_n \alpha_n \epsilon_n$  and  $r_2 = w_+ + \alpha_n \epsilon_n$ . Let  $(\zeta_1, \zeta_2) \in \mathbb{R}^4$  be fixed.

By the estimates of Lemmas 4.7, 4.9 and 4.10, it follows that

$$(4.75) \quad \frac{n^{1/6}}{(2\pi)^{5/2}} \exp \left\{ -\frac{1}{2}(\eta_1^2 + \eta_2^2) + f_n(r_2) - f_n(w_+) - \alpha_n \epsilon_n (a_n \xi_2 - i b_n \eta_2) \right\} \\ \times \iint_{(-\pi/\theta, \pi/\theta) \times \mathbb{R}} F_{r_1, r_2}^{\zeta_1, \zeta_2}(s, t) dt ds \\ = -M_{\sigma_n}(\zeta_1, \zeta_2) + o(1).$$

Inserting into (4.26) and comparing with the kernel  $M_{\sigma_n}$ , results in the expression

$$\begin{aligned} & \tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2) / M_{\sigma_n}(\zeta_1, \zeta_2) \\ &= \exp \left\{ i\tau_n \sqrt{n(1-\tau_n)}(\eta_1 - \eta_2) + \frac{\tau_n n^{-1/3}}{2(1+\tau_n)}(\xi_2^2 - \tau_n \xi_1^2) \right. \\ & \quad \left. + \frac{1}{2}((1-\tau_n^2)\eta_1^2 + (1-\tau_n)\eta_2^2) + \frac{1}{2}\epsilon_n^2 \sigma_n^2 \right\} (1 + o(1)) \\ &= \exp \left\{ i\tau_n \sqrt{n(1-\tau_n)}(\eta_1 - \eta_2) \right\} (1 + o(1)). \end{aligned}$$

Considering the equivalent kernel

$$K_n^{\tau_n'}(\zeta_1, \zeta_2) = \exp\{-i\tau_n \sqrt{n(1-\tau_n)}(\eta_1 - \eta_2)\} \tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2),$$

we see that

$$K_n^{\tau_n'}(\zeta_1, \zeta_2) = M_{\sigma_n}(\zeta_1, \zeta_2)(1 + o(1)) \rightarrow M_{\sigma}(\zeta_1, \zeta_2) \text{ as } n \rightarrow \infty.$$

It remains to prove that  $\tilde{K}_n^{\tau_n}(\zeta, \zeta)$  is dominated, for every sufficiently large  $n$ , by an integrable function on  $(\xi_0, \infty) \times \mathbb{R}$ . This part of the proof is similar to the corresponding step in the proof of the first part of Theorem 2.5 in Section 4.4, although the subtleties of that case are not present here.

Let  $\epsilon < 1$  be fixed and consider any  $n$  so large that

$$(4.76) \quad |\xi_0| < \epsilon n^{1/3}.$$

Choose

$$r_1 = \sqrt{\frac{\tau_n n}{2}}(1 - \epsilon_1)$$

and

$$r_2 = \sqrt{\frac{n}{2\tau_n}}(1 + \epsilon_1),$$

where  $\epsilon_1 = (1 + \sigma_n^2)n^{-1/3}$ . To estimate the integral

$$\iint_{(\theta_n^{-1}\mathbb{T}) \times \mathbb{R}} F_{r_1, r_2}^{\zeta, \zeta}(s, t) dt ds,$$

replace the real line by the contour  $\mathbb{R} + ia_n(1 - \tau_n)\xi/(4\alpha_n)$  when integrating in  $t$ , which is allowed by the condition (4.76) on  $n$ , and put

$$t' = t + i\delta\xi := t + i\frac{a_n(1 - \tau_n)\xi}{4\alpha_n}.$$

This gives

$$(4.77) \quad \left| \iint_{(\theta_n^{-1}\mathbb{T}) \times \mathbb{R}} F_{r_1, r_2}^{\zeta, \zeta}(s, t) dt ds \right| \leq \iint_{(\theta_n^{-1}\mathbb{T}) \times \mathbb{R}} |F_{r_1, r_2}^{\zeta, \zeta}(s, t' - i\delta\xi)| dt' ds \\ \leq \frac{1}{(\sqrt{2\tau_n n \epsilon_1} - \tau_n \alpha_n \delta |\xi_0|)} \int_{\theta_n^{-1}\mathbb{T}} e^{\Phi_{r_1}^{\zeta}(\theta_n s)} ds \int_{\mathbb{R}} e^{\operatorname{Re}(\Xi_{r_2}^{\zeta}(t' - i\delta\xi))} dt'.$$

Using the inequality  $\log(1 + x) \leq x$  provides an estimate

$$(4.78) \quad \int_{\mathbb{R}} e^{\operatorname{Re}(\Xi_{r_2}^{\zeta}(t' - i\delta\xi))} dt' \\ \leq \int_{\mathbb{R}} \exp \left\{ -t'^2 \left( \frac{3\sigma_n^2 + 2 - 2(1 + \sigma_n^2)\epsilon/r_2}{2\tau_n(1 + \epsilon_1)} \right) - \sigma_n \eta t' \right. \\ \left. - \alpha_n \delta (a_n - \alpha_n \delta) \xi^2 + \alpha_n \delta \left( \frac{n}{r_2} - c_n + 2r_2 \right) \xi \right\} dt' \\ \leq C \exp \left\{ \frac{\eta^2}{6} (1 + \epsilon) - \alpha_n \delta (a_n - \alpha_n \delta) \xi^2 + \alpha_n \delta (n/r_2 - c_n + 2r_2) \xi \right\}.$$

Suppose without loss of generality that  $\eta \leq 0$ . Then there is a  $v_\eta \in [0, \pi/2)$  such that  $\Phi_{w-}^{\zeta \prime}(v_\eta) = 0$  and since

$$\begin{aligned} \Phi_{r_1}^{\zeta \prime \prime}(v_\eta) &= 4r_1^2 \cos 2v_\eta - r_1 c_n' \cos v_\eta + r_1 b_n \eta \sin v_\eta \\ &< 4r_1^2 - r_1 c_n' \\ &< -n^{2/3} \sigma_n^2 - \epsilon_1 n (3\tau_n - 1) + \sqrt{n} a_n |\xi_0| \\ &< -n^{2/3} (1 + 3\tau_n \sigma_n^2) < 0, \end{aligned}$$

a saddle point argument shows that

$$\int_{|\theta_n s| < \pi} e^{\Phi_{r_1}^{\zeta}(\theta_n s)} ds \\ \leq C_1 e^{\Phi_{w-}^{\zeta}(v_\eta)} \int_{\mathbb{R}} \exp \left\{ -\frac{(r_1 c_n' - 4r_1^2) \theta_n^2}{2} (s - v_\eta/\theta_n)^2 \right\} ds \leq C e^{\Phi_{w-}^{\zeta}(v_\eta)}.$$

Consider the difference

$$\begin{aligned} \frac{\eta^2}{6} - \Phi_{r_1}^\zeta(v_\eta) &= \frac{1}{6} \left( \eta^2 + 6r_1 b_n \eta \sin v_\eta - 6r_1 c'_n ((\cos v_\eta - 1) + r_1^2(1 - \cos 2v_\eta)) \right) \\ &\geq -\frac{3}{2} b_n^2 r_1^2 \sin^2(v_\eta) - r_1 c'_n (\cos v_\eta - 1) - r_1^2(1 - \cos 2v_\eta) =: g(v_\eta). \end{aligned}$$

Since

$$\begin{aligned} g'(v_\eta) &= r_1^2(4 - 3b_n^2) \sin v_\eta \left( \frac{c'_n}{r_1(4 - 3b_n^2)} - \cos v_\eta \right) \\ &\geq r_1^2(4 - 3b_n^2) \sin v_\eta \left( \frac{c_n - a_n |\xi_0|}{4r_1} - \cos v_\eta \right) \geq 0, \end{aligned}$$

$g$  is increasing and positive on  $[0, \pi/2)$ , which shows that

$$(4.79) \quad \int_{|\theta_n s| < \pi} e^{\Phi_{r_1}^\zeta(\theta_n s)} ds \leq C e^{\eta^2/6}.$$

Recall that  $r_2$  is “almost” a saddle point of  $f_n$ , so

$$\left| \frac{n}{r_2} - c_n + 2r_2 \right| = \left| \frac{\sqrt{2n}\epsilon_1((1 - \tau_n) + \epsilon_1)}{\sqrt{\tau_n}(1 + \epsilon_1)} \right| \leq 6\sqrt{2}(1 + \sigma_n^4)n^{-1/6}.$$

Therefore, by (4.26) and the estimates (4.77), (4.78) and (4.79),

$$\begin{aligned} \tilde{K}_n^{\tau_n}(\zeta_1, \zeta_2) &\leq C_\sigma \exp \left\{ a_n^2(1 - \tau_n) \left( \frac{1}{4(1 + \tau_n)} + \frac{(1 - \tau_n)}{16} - \frac{1}{4} \right) \xi^2 \right\} \\ &\quad \times \exp \left\{ -((1 + \tau_n)(1 + \sigma_n^2) + \alpha_n \delta(n/r_2 - c_n + 2r_2)) \xi \right\} \\ &\quad \times \exp \left\{ -\left( \frac{\tau_n(1 + \tau_n)}{2} - \frac{2 + \epsilon}{6} \right) \eta^2 \right\} \\ &\leq C_{\sigma, \xi_0} e^{-\xi - \eta^2/2}, \end{aligned}$$

which is an integrable function on  $(\xi_0, \infty) \times \mathbb{R}$ . It follows from Lemma (2.1) and the point-wise convergence of  $K_n^{\tau_n'}$  to  $M_\sigma$  that  $\tilde{Z}_n^{\tau_n}$  converges weakly to  $Z_\sigma$  and that the last particle distribution  $F_n^{\tau_n}(t)$  converges to  $F_\sigma(t)$ .  $\square$

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## APPENDIX A.

For sequences of determinantal processes, the following criterion relates weak convergence to convergence of the correlation kernels.

**Lemma A.1.** *For each  $n \geq 1$ , let  $X_n$  be a determinantal point processes on a complete separable metric space  $\Lambda$ , with correlation kernel  $K_n$ . If  $K : \Lambda^2 \rightarrow \mathbb{C}$  is a function such that for every compact set  $A \subseteq \Lambda$ ,*

$$(A.1) \quad \sum_{k=0}^{\infty} \frac{1}{k!} \int_{A^k} \left| \det (K_n(\xi_i, \xi_j))_{i,j=1}^k - \det (K(\xi_i, \xi_j))_{i,j=1}^k \right| d^k \lambda(\xi) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

*then  $K$  is the correlation kernel of a determinantal point process  $X$ , and  $X_n$  converges weakly to  $X$  as  $n \rightarrow \infty$ .*

*Proof.* Let  $\{A_i\}_{i=1}^k$  be a family of bounded disjoint Borel sets in  $\Lambda$  and  $t = (t_1, \dots, t_k) \in [-\ln 2, \infty)^k$  be given. Define the function  $\phi_t(s) = \sum_{j=1}^k (e^{-t_j} - 1) \chi_{A_j}(s)$ . The support of  $\phi$  is the compact set  $A = \overline{\cup_{j=1}^k A_j}$ , and  $|\phi(s)| \leq 1$ . The Laplace transform of the  $k$ -dimensional distributions of  $X_n$  can then be written

$$\begin{aligned} \Psi_n(t) &:= \mathbb{E}_n \left[ \exp \left\{ - \sum_{j=1}^k t_j |X_n \cap A_j| \right\} \right] \\ &= \mathbb{E}_n \left[ \prod_m (1 + \phi_t(x_m)) \right] \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \int_{\Lambda^r} \prod_{j=1}^r \phi_t(\xi_j) \rho_r^n(\xi_1, \dots, \xi_r) d^r \lambda(\xi) \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \int_{\Lambda^r} \prod_{j=1}^r \phi_t(\xi_j) \det (K_n(\xi_i, \xi_j))_{i,j=1}^r d^r \lambda(\xi) \\ &\rightarrow \sum_{r=0}^{\infty} \frac{1}{r!} \int_{\Lambda^r} \prod_{j=1}^r \phi_t(\xi_j) \det (K(\xi_i, \xi_j))_{i,j=1}^r d^r \lambda(\xi) \end{aligned}$$

as  $n \rightarrow \infty$ , by (2.2) and the hypothesis. Convergence of the Laplace transforms for all  $t$  in a neighbourhood of the origin implies convergence of the joint distributions. The existence of a determinantal point process with correlation kernel  $K$  is an immediate consequence of the fact that the  $K_n$  all are correlation kernels; the consistency conditions e.g. in [13] are easily seen to be satisfied.  $\square$

**Lemma A.2.** *Let  $Z$  be a determinantal point process on  $\mathbb{R}^2$  with a Hermitian correlation kernel  $K$  and suppose that*

$$\int_{(\xi_0, \infty) \times \mathbb{R}} K(\zeta, \zeta) d\zeta < \infty$$

for every  $\xi_0 \in \mathbb{R}$ . Then  $Z$  has a last particle almost surely and the distribution function  $F$  of the last particle is given by

$$(A.2) \quad F(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_{((t,\infty) \times \mathbb{R})^r} \det(K(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta.$$

If  $K$  defines a locally trace class integral operator on  $L^2(\mathbb{R}^2)$ ,  $F(t)$  can be more compactly expressed as a Fredholm determinant

$$(A.3) \quad F(t) = \det(I - K)_{L^2((t,\infty) \times \mathbb{R})}.$$

*Proof.* Let  $t \in \mathbb{R}$  be given and suppose  $s > t$ . Applying (2.2), with  $\phi(s, t) = \chi_{((t,s) \times (-s,s))}$ , gives

$$\begin{aligned} \mathbb{P}[|Z \cap ((t,s) \times (-s,s))| = 0] \\ = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_{((t,s) \times (-s,s))^r} \det(K(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta. \end{aligned}$$

Because  $K$  is Hermitian, the matrix  $(K(\zeta_i, \zeta_j))_{i,j=1}^r$  is positive definite, so by Hadamard's inequality

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{1}{r!} \int_{((t,\infty) \times \mathbb{R})^r} \det(K(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta &\leq \sum_{r=0}^{\infty} \frac{1}{r!} \left( \int_{(t,\infty) \times \mathbb{R}} K(\zeta, \zeta) d\zeta \right)^r \\ &= \exp \left\{ \int_{(t,\infty) \times \mathbb{R}} K(\zeta, \zeta) d\zeta \right\} < \infty. \end{aligned}$$

Since correlation functions are non-negative,

$$\begin{aligned} \left| \frac{(-1)^r}{r!} \int_{((t,s) \times (-s,s))^r} \det(K(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta \right| \\ \leq \frac{1}{r!} \int_{((t,\infty) \times \mathbb{R})^r} \det(K(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta, \end{aligned}$$

so by the dominated convergence theorem

$$\begin{aligned} F(t) &= \lim_{s \rightarrow \infty} \mathbb{P}[|Z \cap ((t,s) \times (-s,s))| = 0] \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_{((t,\infty) \times \mathbb{R})^r} \det(K(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta. \end{aligned}$$

□

*Proof of Lemma 2.1.* We begin with the convergence of the last particle distribution. The existence and distribution of a last particle for  $Z_n$  is given by Lemma A.2. By Hadamard's inequality,

$$\left| \frac{(-1)^r}{r!} \det(K_n(\zeta_i, \zeta_j))_{i,j=1}^r \right| \leq \frac{1}{r!} \prod_{i=1}^r K_n(\zeta_i, \zeta_i) \leq \frac{1}{r!} \prod_{i=1}^r B(\zeta_i).$$

Now

$$\int_{((t,\infty)\times\mathbb{R})^r} \frac{1}{r!} \prod_{i=1}^r B(\zeta_i) d^r \zeta = \frac{1}{r!} \left( \int_{(t,\infty)\times\mathbb{R}} B(\zeta) d\zeta \right)^r = \frac{C_t^r}{r!}$$

and

$$\sum_{r=0}^{\infty} \frac{C_t^r}{r!} = e^{C_t} < \infty,$$

so applying the dominated convergence theorem twice gives the conclusion

$$\begin{aligned} \lim_{n \rightarrow \infty} F_n(t) &= \lim_{n \rightarrow \infty} \sum_{r=0}^{\infty} \int_{((t,\infty)\times\mathbb{R})^r} \frac{(-1)^r}{r!} \det(K_n(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta \\ &= \sum_{r=0}^{\infty} \int_{((t,\infty)\times\mathbb{R})^r} \lim_{n \rightarrow \infty} \frac{(-1)^r}{r!} \det(K_n(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta \\ &= \sum_{r=0}^{\infty} \int_{((t,\infty)\times\mathbb{R})^r} \frac{(-1)^r}{r!} \det(K(\zeta_i, \zeta_j))_{i,j=1}^r d^r \zeta \\ &= F(t). \end{aligned}$$

To verify the condition for weak convergence of point processes stated in Lemma A.1, the same argument applies since an arbitrary compact set  $A \subset \mathbb{R}^2$  is contained in a set of the form  $(t, \infty) \times \mathbb{R}$ .  $\square$

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